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The general commutant extension problem

Interpolatieproblemen kunnen ogenschijnlijk sterk van elkaar verschillen. Veel interpolatieproblemen kunnen worden gezien als speciale gevallen van het zogenaamde probleem van de commuterende uitbreiding. In dit artikel presenteert Ciprian Foias een overzicht van de huidige stand van zaken omtrent dit fundamentele onderwerp van de operatorentheorie. Op dit terrein heeft hij belangrijke bijdragen geleverd. Het onderhavige artikel is gebaseerd op Foias' lezing gegeven aan de Universiteit Leiden, ter ere van zijn benoeming tot Stieltjes Visiting Professor 2000.

The topic of this paper is an open question in Operator Theory which can be traced back to Stieltjes' solution to the moments problems in [22], and which touches several fields of pure and applied Functional Analysis.

Statement of the problem

The General Commutant Extension Problem (GCEP) is formulated as follows (see [10], [7]). Let G and H be (complex) Hilbert spaces and let $S : G \to G$ and $T : H \to H$ be two bounded linear operators on G and H, respectively. Let G_0 be a (closed linear) subspace of G, invariant under S and let S_0 be the restriction $S|G_0$ of S to G_0. Let moreover $A_0 : G_0 \to H$ be a bounded linear operator, such that

$$A_0 S_0 = T A_0. \tag{1}$$

When does there exist a bounded linear operator $A : G \to H$, such that

$$A|G_0 = A_0 \tag{2a}$$

(i.e. extending A_0), and

$$AS = TA? \tag{2b}$$

In case such an operator A exists, find

$$\alpha = \min \{|A|| : A \text{ satisfies (2a) and (2b)}\}. \tag{3}$$

Clearly, without loss of generality we can assume

$$||S||, ||T|| \leq 1. \tag{4}$$

Above, $||\cdot||$ stands for the operator norm, and as throughout, enough familiarity with the basic concepts in Operator Theory (e.g. as in [15]) is surmized so that further detailed definitions are not deemed necessary. We observe that a solution A to the problem above does not always exist, as illustrated by the following elementary example [2].

$$G = H = \mathbf{C}^2, \qquad G_0 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{C}^2 : x_2 = 0 \right\} \tag{5a}$$

and

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_0 x \equiv x \in G_0. \tag{5b}$$

Then (1) holds, but if (2a) holds too, then

$$A = \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix} \text{ and } AS = \begin{bmatrix} 0 & 1 \\ 0 & a_{22} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix} = TA. \quad (6)$$

The worth of any open problem in mathematics depends, in particular, on the answers one provides to the following questions:

1. Is the problem a hard nut to crack in its own field?
2. Is the problem relevant to other fields of mathematics?
3. Can the solution to the problem be of use in scientific fields beyond pure mathematics?
4. Is there any hope to find a 'good' answer to the problem?

I will present some answers to these questions. A first answer to question 1 is easy to give: for me and some of my collaborators and competitors in Operator Theory, the answer is "yes"; otherwise we would have found a satisfactory solution in the last eight years since we became interested in the problem. The difficulty resides in aiming to give an affirmative answer to both questions 3 and 4; a thorough discussion of the latter question will be given in the last part of the paper. The answers to questions 2 and 3 is twice an emphatic "yes". We will devote the main part of the paper to provide a brief but hopefully instructive argument for this statement.

Interpolation and the GCEP

At the beginning of the past century, Carathéodory considered the following interpolation problem [3]: When does there exist an analytic function $f(z)$ in the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ with Taylor expansion

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + O(z^{n+1}), \quad (6a)$$

where $a_0, \ldots, a_n \in \mathbf{C}$ are a priori given, and such that

$$\Re f(z) \geq 0 \quad \text{for } z \in \mathbf{D}? \quad (6b)$$

Notice that

$$a_k = \frac{1}{2\pi r^k} \int_0^{2\pi} f(re^{it}) e^{-ikt} dt \quad (0 < r < 1,\ k = 0, 1, \ldots, n),$$

so that the problem can be construed as a partial version of Stieltjes's moments problem. Note also that replacing this by

$$f(z) := (f(z) - 1)/(f(z) + 1), \quad (z \in \mathbf{D}), \quad (0)$$

we can consider the equivalent problem, considered by Schur [21], where the condition (6b) is replaced by

$$|f(z)| \leq 1 \quad \text{for } z \in \mathbf{D}. \quad (6c)$$

Schur found the following remarkable solution. If f satisfying (6a) and (6c) exists, then $|a_0| = |f(0)| \leq 1$. If $|a_0| = 1$, then the maximum principle forces $f(z) \equiv a_0$. So if $a_j = 0$, $1 \leq j \leq n$, we have exactly one solution. If not, there is no solution. If $|a_0| < 1$, then

$$\frac{f(z) - a_0}{1 - \bar{a}_0 f(z)} = z g(z) \quad (z \in \mathbf{D}) \quad (7a)$$

where $g(z)$ is analytic in $\mathbf{D}$,

$$g(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} + O(z^n), \quad (7b)$$

$$|g(z)| \leq 1 \quad (z \in \mathbf{D}) \quad (7c)$$

and $b_0 = a_1(1 - |a_0|^2)^{-1}$,

$$b_j = \left(a_{j+1} + \bar{a}_0 \sum_{i=1}^{j} b_{j-i} a_i \right) \Big/ (1 - |a_0|^2) \quad (7d)$$

for $j = 1, 2, \ldots, n-1$. This is exactly the same problem but with n replaced by $n-1$. Denote $c_0 = a_0$, $c_1 = b_0$. Let

$$n := n - 1,\ a_j := b_j,\ 0 \leq j \leq n-1;$$

define b_j as in (7d) and $c_2 := b_0$. If $|c_2| = 1$, check if $b_j = 0$, $1 \leq j \leq n-1$. If not, no solution exists. If $|c_2| < 1$, continue.

This algorithm either stops at the first $c_j \in \overline{\mathbf{D}} \setminus \mathbf{D}$ (where $j \leq n$) or yields the coefficients $c_0, c_1, \ldots, c_n$ in $\mathbf{D}$. In the first case the problem either has exactly one solution or none, while in the second case, the solution is an explicit rational function in z and an arbitrary analytic function $g_{\text{last}}(z)$ in $\mathbf{D}$ satisfying (7c).

These Schur coefficients $c_0, c_1, \ldots$ have a remarkable geophysical interpretation [18] (see also [8]). Indeed, consider a horizontally multilayered (isotropic linear elastic) medium such that the top layer (0-layer) and the bottom layer ($(n+1)$-layer) are of infinite width and all contacts along the interfaces are welded. We also consider vertically moving horizontal primary waves (i.e. oscillating in the vertical direction only). At each interface, say the j-interface, a downgoing wave of amplitude $D_j(t)$ will produce an upgoing (i.e. reflected) wave with amplitude $r_j D_j(t)$ and a downgoing wave (i.e. transmitted through the interface) with amplitude $(1 + r_j) D_j(t)$. The coefficient r_j is called the reflection coefficient of the j-interface and $-1 \leq r_j \leq 1$. In fact in nature $|r_j| = 1$ is a very rare occurrence. So we will consider only the case when $-1 < r_j < 1$ for all $j = 0, 1, \ldots, n$. For upgoing waves the same holds true but with r_j replaced by $-r_j$. For simplicity, we shall assume that the j-layers (for $j = 1, 2, \ldots, n$) are crossed by these waves in .5 units of time. It can be easily proven (see [8], Ch. XVII) that in this case, by producing a vertical oscillation $D(t)$ on the upper side of the 0-interface, one will register at the same site an upgoing wave $V(t)$ of the form

$$V(t) = \sum_{j=0}^{\infty} a_j D(t - j)$$

such that the Schur coefficients $c_0, c_1, \ldots, c_n$ associated to the Carathéodory-Schur problem (3.1a), (3.1c) are precisely the reflexion coefficients $r_0, r_1, \ldots, r_n$. Moreover the function

$$f(z) = a_0 + a_1 z_0 + \cdots \quad (8)$$

is exactly the solution to this problem when $g_{\text{last}}(z) \equiv 0$.

What does the above discussion have to do with the problem formulated at the beginning? In fact, the hidden mathematical background of the last paragraph is an affirmative "good" solution to a particular case of the General Commutant Extension Problem (referred to as GCEP in the sequel), namely

Theorem 1. *If in GCEP, $||S|| \leq 1$ and T^* is an isometry (i.e. $||T^* h|| = ||h||$ for all $h \in H$), then α in formula (3) equals $||A_0||$.*

This result is due to Sarason [20] and (in the general form) to Sz.-Nagy et al.[23–24], and is referred to in the literature as the Commutant Lifting Theorem, a name coined by Douglas, Muhly and Pearcy [6]. To see that the Carathéodory-Schur problem (6a), (6c) can be solved by applying Theorem 1, define

$$G = H = \ell^2_+ := \\ := \{x = [x_0, x_1, \ldots] : x_j' s \in \mathbf{C}, ||x||^2 = \sum_{j=1}^{\infty} |x_j|^2 < \infty\}, \tag{9a}$$

$$G_0 = \{x = [x_0, x_1, \ldots] \in \ell^2_+ : x_{n+1} = x_{n+2} = \cdots = 0\}, \tag{9b}$$

$$Tx = Sx = S[x_0, x_1, x_2, \ldots] = [x_1, x_2, \ldots] \text{ for } x \in G \tag{9c}$$

$$A_0 x = \begin{bmatrix} \bar a_0 & \bar a_1 & \cdots & \bar a_n & & & \\ & \bar a_0 & \bar a_1 & \cdots & \bar a_n & & \\ & & \bar a_0 & \bar a_1 & \cdots & \bar a_n & \\ & & & \bar a_0 & \bar a_1 & \cdots & \ddots \\ & & & & \ddots & \ddots & \end{bmatrix} x \text{ for } x \in G_0. \tag{9d}$$

Note that

$$||A_0|| = \left|\left|\begin{bmatrix} \bar a_0 & \cdots & \bar a_n \\ & \bar a_0 & \vdots \\ & & \bar a_0 \end{bmatrix}\right|\right| = \left|\left|\begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & a_0 & \\ a_n & \cdots & a_1 & a_0 \end{bmatrix}\right|\right|,$$

where the last two norms are those of the corresponding matrices as operators on $\mathbf{C}^{n+1}$. Due to Theorem 1, there exists an $A = \ell^2_+ \mapsto \ell^2_+$ satisfying $||A|| = ||A_0||$ and (2a,b). These last two conditions force

$$Ax = \begin{bmatrix} \bar a_0 & \bar a_1 & \cdots & \bar a_n & \bar a_{n+1} & \cdots & & \\ & \bar a_0 & \bar a_1 & \cdots & \bar a_n & \bar a_{n+1} & \cdots & \\ & & \bar a_0 & \bar a_1 & \cdots & \bar a_n & \bar a_{n+1} & \cdots \\ & & & \ddots & \ddots & & \ddots & \ddots \end{bmatrix} x \text{ for } x \in \ell^2_+$$

and, setting $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ (for $z \in \mathbf{D}$) we have

$$\sup\{|f(z)| : z \in \mathbf{D}\} = ||A|| = ||A_0||. \tag{9d}$$

This is Sarason's way to obtain an optimal solution to the Carathéodory-Schur problem ([26]; see also [8], Ch. X).

Good answers

Theorem 1 turned out to be quite useful to Control Theory. For theoretical guidance and for dimension-free algorithms, especially when transcendental transfer functions cannot be avoided, see [8], Ch. XII or [9], Ch. VII. For a much more engineering oriented approach involving Theorem 1, see [12]. In nonlinear analytic Control Theory as developed in [13], [14] and [11]one has to replace Theorem 1 with a rather complicated causal version of it. It turned out (see [10]) that this version is equivalent to a new particular case of the GCEP, namely

$$S \text{ is isometric;} \tag{10a}$$

$$\{g \in G : \exists n = n(g) \text{ such that } S^n g \in G_0\} \text{ is dense in } G. \tag{10b}$$

The problem has a 'good' solution only in particular cases [10], [17], for instance:

Theorem 2. [10] *If (10a,b) hold and* $\ker T := \{h \in H : Th = 0\} = \{0\}$*, then GCEP has a solution if and only if there exists a constant* $M < \infty$ *such that*

$$||P_{G_n} S^{*n} A_0^* h|| \le M ||T^{*n} h|| \tag{10c}$$

for all $h \in H, n = 0, 1, 2, \ldots$ *where* P_{G_n} *is the orthogonal projection of* G *onto* $G_n = \{g \in G : S^n g \in G_0\}$*. In this case (see (3))*

$$\alpha = \min\{M : M \text{ as in (10c)}\}. \tag{10d}$$

Another remarkable particular case in which GCEP has a 'good' answer is the following

Theorem 3. [4–5] *Let S and T be isometries. Then GCEP has a solution if and only if there exists a constant* $M < \infty$ *such that*

$$||A_0|| \le M, \tag{11a}$$

$$||P_{\ker T^{*n}} A_0 g_0|| \le M ||P_{\ker S^{*n}} g_0|| \tag{11b}$$

for all $g_0 \in G_0, n = 0, 1, \ldots$ *In this case*

$$\alpha = \min\{M : M \text{ as in (11a), (11b)}\}. \tag{11c}$$

In order to give a different type of example, let

$$S_c := P_{G_c} S | G_c, \tag{12a}$$

where $G_c = G \ominus G_0$ is the subspace of G formed by all vectors orthogonal on G_0; equivalently,

$$S_c^* = S^* | G_c. \tag{12b}$$

As before, the vertical bar in (12a), (12b) denotes the restriction to the space following the bar, P_{G_c} denotes the orthogonal projection of G onto G_c. Now we can formulate the following particular answer to GCEP:

Theorem 4. [10] *If*

$$\lim_{n \to \infty} ||S_c^n g|| \ge \theta ||g|| \text{ for } g \in G_c \tag{13}$$

for some $\theta \in (0,1)$*, then the* α *in (3) exists and satisfies*

$$\alpha \le (1 + \theta^2)^{1/2} \theta^{-1} ||A_0||. \tag{14}$$

Note that condition (10b) can be reformulated in terms of S_c as follows:

$$\bigcup_{n=0}^{\infty} \ker S_c^n \text{ is dense in } G_c, \tag{15}$$

which is the extreme counterpart of (13).

The Commutant Lifting Theorem (i.e., Theorem 1) has two intriguing generalizations, both of which can be viewed as 'good' answers to the GCEP in appropriate cases. The first one is given by the following remarkable results of Treil and Volberg, reformulated in the following

Theorem 5. [25] *If in the GCEP,* $||S|| \le 1$ *and* $||T^* h|| \ge ||h||$ *for all* $h \in H$*, then* $\alpha = ||A_0||$.

The second generalization of Theorem 1 (in fact of Theorem 5 as well) is

Theorem 6. [2] *If in the GCEP, $||S|| \le 1$ and there exists an operator Ω on H such that*

$$\Omega \ge 0 \text{ (i.e. } (\Omega h, h) \ge 0 \tag{16}$$

for all $h \in H$), $T\Omega T^* - \Omega \ge 0$ and $\Omega - A_0 A_0^* \ge 0$, then there exists an A satisfying (1.2a,b) and

$$AA^* \le \Omega. \tag{17}$$

Thus in particular

$$\alpha \le ||\Omega||^{1/2}. \tag{18}$$

It is clear that Theorem 6 implies Theorem 5 (take $\Omega = I$) and that Theorem 5 implies Theorem 1 (obviously, T^* is an isometry if it satisfies $||T^*h|| = ||h||$ for all $h \in H$).

Moreover, the original proof of Theorem 3 was obtained by using Theorem 1, i.e., Theorem 1 implies Theorem 3. The latter can be given also a direct proof (see [9], Ch. XII). Theorem 2 and 4 are quite independent of the other theorems. It is a strange fact (from [1]) that the GCEP reduces to its following particular case:

$$S \text{ is an isometry} \tag{19a}$$

(in fact such that $||S^{*n}g|| \to 0$ for all $g \in G$) and

$$\bigcup_{n=1}^{\infty} \ker T^{*n} \text{ is dense in } H. \tag{19b}$$

Moreover, in this case (11b) is still a necessary condition (albeit not sufficient!) for the existence of A in the GCEP.

A disturbing aspect concerning the illustrative list of answers to particular cases of the GCEP provided by Theorems 1–6 is that they are so different that it is hard to imagine one simple general answer to the GCEP which will trivialize these cases. In Von Neumann's opinion, when any portion of mathematics starts looking rather like the delta of a river and no more like the river itself, that may be translated from visual to auditive perception as the swan's last song. In a plain formulation: The General Commutant Extension Problem may not have a 'good' solution at all.

So we should now discuss what is a 'good' solution to the GCEP. However, it is much easier to show what is a 'bad' solution. First remark that to solve the GCEP means to find an explicit way to determine when α in (3) exists and, if so, to compute it. There may be very many ways to do that. For instance, in the next paragraph we will present such a way (see below, Theorem 8), due to C. Gu and myself ([7], § 11). However, this almost complete solution to the GCEP is a good example of a 'bad' answer to the GCEP. For instance, it does not seem to provide a new approach to any of the Theorems 1–6. In fact, it just seems useless.

Bad solutions

In this paragraph (which can be skipped by a casual reader) we will collect some simple facts necessary to prove Theorem 8.

Lemma 1. *The GCEP has a solution A if and only if there exists a bounded linear operator $X : G_c \to H$ such that*

$$TX - XS_c = A_0 P_{G_0} S | G_c. \tag{20a}$$

In this case

$$A = A_0 P_{G_0} + X P_{G_c}. \tag{20b}$$

The proof is pure elementary algebra once one notices that

$$P_{G_c} S = P_{G_c} S_c. \tag{20c}$$

As an aside one has the following corollary.

Theorem 7. *If the spectra of T and S_c are disjoint, then the GCEP has a unique solution A.*

Indeed in this case (20a) has a unique solution X (see for instance [19]).

Lemma 2. *Let X be a Banach space, $\Gamma : X \to X$ a bounded linear operator on X and $f \in X^*$ a bounded linear functional on X. Then*

$$f \in \Gamma^* X^* \tag{21a}$$

if and only if there exists a constant $M < \infty$ such that

$$|f(x)| \le M||\Gamma x|| \quad \text{for all } x \in X. \tag{21b}$$

It is clear that if $f = \Gamma^* g$ with some $g \in X^*$, then (21b) holds with $M = ||g||$. For constructing such a g, if (21b) holds, apply the Hahn-Banach theorem to the functional h defined on ΓX by $h(\Gamma x) = f(x)$, $x \in X$.

Denote by X the space $\sigma_1(H, G_c)$ of all finite trace operators $F : H \to G_c$, i.e. such that F^*F has the form

$$F^*Fh = \sum_{j=1}^{\infty} \lambda_j f_j (h, f_j) \text{ for all } h \in H, \tag{22a}$$

where $\{f_j\}_{j=1}^{\infty}$ is an orthonormal system in H and

$$||F||_1 = \sum_{j=1}^{\infty} \lambda_j^{1/2} < \infty. \tag{22b}$$

The map $F \to ||F||_1$ is then a norm on X and X endowed with this norm is a Banach space (see [16], Ch. VI). The dual space X^* of X can then be identified with the space $B(G_c, H)$ of all bounded linear operators $Y : G_c \to H$ endowed with the usual operator norm. The duality is given by

$$\langle Y, F\rangle = \text{trace}(FY) \text{ for } F \in X, Y \in X^*. \tag{22c}$$

Now define the operator Γ on X by

$$\Gamma F = FT - S_c F \text{ for } F \in X. \tag{23a}$$

Then

$$\Gamma^* Y = TY - YS_c \text{ for } Y \in X^*. \tag{23b}$$

Thus Lemmas 1 and 2 have the following immediate consequence:

Theorem 8. *The GCEP has a solution A if and only if there exists a constant $M < \infty$ such that*

$$|\text{trace}(A_0 P_{G_0} SF)| \le M||FT - S_c F||_1 \text{ for all } F \in \sigma_1(H, G_c). \tag{24}$$

Note that in view of (20b) and the proof of Lemma 2, we also have

$$\alpha \le \sqrt{2} \max\{||A_0||, \min\{M : M \text{ as in (24)}\}\}. \tag{25}$$

Conclusion

As already mentioned above, I do not know any proof for any of the Theorems 1–6 based on Theorem 8. But there are other reasons for which I consider this theorem to provide a 'bad' solution to the GCEP. They will be evident if I conclude with my definition of what should be a 'good' solution to the GCEP. Namely, first find a linear operator $\hat{A}_0$ defined on a specified dense linear subspace of an adequate Hilbert space with values in (perhaps another) Hilbert space, $\hat{A}_0$ being uniquely determined by the data, such that A exists if and only if A_0 is bounded and in this case $\alpha = ||\hat{A}_0||$. Secondly, deduce all Theorems 1–6 by estimating directly the norm $||A_0||$. Thirdly, obtain a workable algorithm for computing $||A_0||$ in the case (10a,b). So I conclude wishing good luck to anyone who will try to find such a 'good' solution to the GCEP.

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