Pentti Haukkanen
Department of Mathematics, Statistics and Philosophy
FIN-33014 University of Tampere, Finland
mapehau@uta.fi

An elementary linear algebraic approach to even functions (mod r)

An arithmetical function $f(n)$ is said to be even $(\mathrm{mod}\, r)$ if $f(n) = f((n,r))$ for all $n \in \mathbf{Z}^+$, where (n,r) is the greatest common divisor of n and r. It is well known that every even function f $(\mathrm{mod}\, r)$ possesses a representation $f(n) = \sum_{d|r} \alpha(d)C(n,d)$ in terms of Ramanujan's sum $C(n,r)$. In this paper we interpret two known expressions of α using elementary linear algebra. We also present the known convolution representation of an even function $(\mathrm{mod}\, r)$ in a simple way using the Dirichlet convolution.

Introduction

Let r be a fixed positive integer greater than 1. An arithmetical function $f(n)$ is said to be *even* $(\mathrm{mod}\, r)$ if

$$f(n) = f((n,r))$$

for all $n \in \mathbf{Z}^+$, where (n,r) is the greatest common divisor of n and r.

The concept of an even function $(\mathrm{mod}\, r)$ originates with Cohen [1]. Even functions $(\mathrm{mod}\, r)$ were further studied by Cohen in subsequent papers [2, 3, 4]. General accounts of even functions $(\mathrm{mod}\, r)$ can be found, for example, in the books by McCarthy [5] and Sivaramakrishnan [6].

It is well known that every even function f $(\mathrm{mod}\, r)$ possesses a representation

$$f(n) = \sum_{d|r} \alpha(d)C(n,d)$$

in terms of Ramanujan's sum $C(n,r)$, where

$$C(n,r) = \sum_{\substack{k\ (\mathrm{mod}\, r)\\ (k,r)=1}} \exp(2\pi ikn/r).$$

The coefficients $\alpha(d)$, where $d|r$, are unique and are referred to as the Fourier coefficients of f. It is well known that

$$\alpha(d) = r^{-1}\sum_{e|r} f(r/e)C(r/d,e) = (r\phi(d))^{-1}\sum_{a\ (\mathrm{mod}\, r)} f(a)C(a,r),$$

where ϕ is Euler's totient function. In this paper we interpret these expressions of α using elementary linear algebra. We also present the known convolution representation of an even function $(\mathrm{mod}\, r)$ in a simple way using the Dirichlet convolution. Some examples of even functions $(\mathrm{mod}\, r)$ are also given.

The vector space of even functions (mod r)

Let r (≥ 2) be fixed. Let E_r denote the set of all even functions $(\mathrm{mod}\, r)$.

Theorem 1. *The set E_r forms a complex vector space under the usual sum of functions and the scalar multiplication.*

The proof is an easy exercise in elementary linear algebra.

Theorem 2. *The dimension of the vector space E_r is $\tau(r)$, the number of positive divisors of r.*

Proof. Let $d_1, d_2, \ldots, d_{\tau(r)}$ be the positive divisors of r in ascending order. For each divisor d_i, $i = 1, 2, \ldots, \tau(r)$, define the function $\rho_r^{(i)}$ as

$$\rho_r^{(i)}(n) = \begin{cases} 1 & \text{if } (n,r) = d_i, \\ 0 & \text{otherwise.} \end{cases}$$

We prove that $\{\rho_r^{(i)} : i = 1, 2, \ldots, \tau(r)\}$ is a basis of the vector space E_r. Clearly $\rho_r^{(i)} \in E_r$ for all $i = 1, 2, \ldots, \tau(r)$. Every $f \in E_r$ can be written as a linear combination of the functions $\rho_r^{(i)}$, $i = 1, 2, \ldots, \tau(r)$, as

$$f(n) = f(d_1)\rho_r^{(1)}(n) + f(d_2)\rho_r^{(2)}(n) + \cdots + f(d_{\tau(r)})\rho_r^{(\tau(r))}(n).$$

Further, it can be verified that the functions $\rho_r^{(i)}$, $i = 1, 2, \ldots, \tau(r)$, are linearly independent. This shows that $\{\rho_r^{(i)} : i = 1, 2, \ldots, \tau(r)\}$ is a basis of the vector space E_r and thus the dimension of the vector space E_r is $\tau(r)$. □

The inner product space of even functions (mod r)

The Dirichlet convolution of arithmetical functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Theorem 3. *Let r (≥ 2) be fixed. The vector space E_r forms a complex inner product space with*

$$\langle f, g\rangle = \sum_{d|r} \phi(d)f(r/d)\overline{g}(r/d) = (\phi * f\overline{g})(r), \tag{1}$$

where $\overline{g}(n) = \overline{g(n)}$, the complex conjugate of $g(n)$.

The proof is an easy exercise in elementary linear algebra.

Lemma 1. [5, p. 79] *Let $d_1 \mid r$ and $d_2 \mid r$. Then*

$$\sum_{e\mid r} C(r/e,d_1)C(r/e,d_2)\phi(e) = \begin{cases} r\phi(d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. If $d \mid r$, then $C(n,d)$ is even $(\bmod r)$.

Theorem 4. *The following set is an orthonormal basis of the inner product space E_r:*

$$\{(r\phi(d))^{-\frac{1}{2}}C(\cdot,d) : d \mid r\}. \tag{2}$$

Proof. As the dimension of the inner product space E_r is $\tau(r)$ and the number of elements in the set (2) is $\tau(r)$, it suffices to show the set (2) is an orthonormal subset of E_r. This follows easily from the above remark and Lemma 1. □

Theorem 5. *An arithmetical function f is even $(\bmod r)$ if and only if it has a representation*

$$f(n) = \sum_{d\mid r} \alpha(d)C(n,d), \tag{3}$$

where

$$\alpha(d) = r^{-1}\sum_{e\mid r} f(r/e)C(r/d,e). \tag{4}$$

Proof. If f possesses the representation (3), then f is a linear combination of even functions $(\bmod r)$. As even functions $(\bmod r)$ form a vector space, this linear combination is even $(\bmod r)$. Suppose that f is even $(\bmod r)$. Then by Theorem 4

$$f(n) = \sum_{d\mid r}\langle f,(r\phi(d))^{-\frac{1}{2}}C(\cdot,d)\rangle (r\phi(d))^{-\frac{1}{2}}C(n,d), \tag{5}$$

where

$$\langle f,(r\phi(d))^{-\frac{1}{2}}C(\cdot,d)\rangle = \sum_{e\mid r}\phi(e)f(r/e)(r\phi(d))^{-\frac{1}{2}}C(r/e,d).$$

It is known [5, p. 93] that $\phi(e)C(r/e,d) = \phi(d)C(r/d,e)$. Therefore

$$\langle f,(r\phi(d))^{-\frac{1}{2}}C(\cdot,d)\rangle = r^{-\frac{1}{2}}\phi(d)^{\frac{1}{2}}\sum_{e\mid r} f(r/e)C(r/d,e). \tag{6}$$

Combining formulas (5) and (6) gives formula (3). □

The Cauchy product

Let f and g be even functions $(\bmod r)$. Their *Cauchy product* is defined as

$$(f\circ g)(n) = \sum_{a+b\equiv n\,(\bmod r)} f(a)g(b) = \sum_{a(\bmod r)} f(a)g(n-a).$$

Lemma 2. [5, p. 76] *Let $d_1, d_2 \mid r$. Then*

$$\sum_{a+b\,\equiv n\,(\bmod r)} C(a,d_1)C(b,d_2) = \begin{cases} rC(n,d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6. *Let f and g be even functions $(\bmod r)$ with Fourier coefficients, respectively, $\alpha_f(d)$ and $\alpha_g(d)$, $d \mid r$. Then $f\circ g$ is even $(\bmod r)$ with Fourier coefficients $r\alpha_f(d)\alpha_g(d)$, $d\mid r$.*

Proof. By Theorem 5

$$\begin{aligned}(f\circ g)(n) &= \sum_{a+b\equiv n(\bmod r)}\sum_{d\mid r}\alpha_f(d)C(a,d)\sum_{e\mid r}\alpha_g(e)C(b,e)\\ &= \sum_{d\mid r}\sum_{e\mid r}\alpha_f(d)\alpha_g(e)\sum_{a+b\equiv n(\bmod r)}C(a,d)C(b,e).\end{aligned}$$

By Lemma 2 $(f\circ g)(n) = \sum_{d\mid r} r\alpha_f(d)\alpha_g(d)C(n,d)$. Thus $f\circ g$ is even $(\bmod r)$ with Fourier coefficients $r\alpha_f(d)\alpha_g(d)$, $d\mid r$. □

Theorem 7. *Inner product (1) can be written as $\langle f,g\rangle = (f\circ\overline{g})(0)$.*

Proof. By Parseval's identity

$$\langle f,g\rangle = \sum_{d\mid r}\langle f,(r\phi(d))^{-\frac{1}{2}}C(\cdot,d)\rangle\overline{\langle g,(r\phi(d))^{-\frac{1}{2}}C(\cdot,d)\rangle}.$$

By equations (6) and (4)

$$\begin{aligned}\langle f,g\rangle &= r\sum_{d\mid r}\alpha_f(d)\overline{\alpha}_g(d)\phi(d)\\ &= r\sum_{d\mid r}\alpha_f(d)\overline{\alpha}_g(d)C(0,d) = (f\circ\overline{g})(0).\end{aligned}$$

This completes the proof of Theorem 7. □

Theorem 8. *The Fourier coefficients of an even function f $(\bmod r)$ have the expression*

$$\alpha(d) = (r\phi(d))^{-1}\sum_{a\,(\bmod r)} f(a)C(a,r). \tag{7}$$

Proof. According to equation (5) and Theorem 7

$$\begin{aligned}f(n) &= \sum_{d\mid r}\sum_{a+b\equiv 0\,(\bmod r)} f(a)(r\phi(d))^{-\frac{1}{2}}C(b,d)(r\phi(d))^{-\frac{1}{2}}C(n,d)\\ &= \sum_{d\mid r}\Big\{(r\phi(d))^{-1}\sum_{a\,(\bmod r)} f(a)C(-a,d)\Big\}C(n,d).\end{aligned}$$

As $C(-a,d) = C(a,d)$, we obtain (7). □

A convolution representation

Theorem 9. *An arithmetical function f is even $(\bmod r)$ if and only if it can be written as*

$$f(n) = \sum_{d\mid(n,r)} g(d), \tag{8}$$

where g is an arithmetical function (which may depend on r). In this case $g = f*\mu$, where μ is the Möbius function.

Proof. Let f be even $(\bmod r)$. As μ is the inverse of the constant function 1 under the Dirichlet convolution, we have

$$f(n) = f((n,r)) = \sum_{d\mid(n,r)}(f*\mu)(d).$$

Thus f possesses (8) with $g = f * \mu$. Conversely, if (8) holds, then

$$f(n) = \sum_{d|(n,r)} g(d) = \sum_{d|((n,r),r)} g(d) = f((n,r)).$$

Thus f is even $(\mathrm{mod}\, r)$. □

Lemma 3. [5, p. 71] *We have*

$$\sum_{d\,|r} C(n,d) = \begin{cases} r & \text{if } r|n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 10. *The Fourier coefficients of an even function* f $(\mathrm{mod}\, r)$ *have the expression:*

$$\alpha(d) = r^{-1} \sum_{e|r/d} (f * \mu)(r/e)e. \tag{9}$$

Proof. Let $g = f * \mu$. We put the coefficients (9) on the right-hand side of (3) to get

$$\sum_{d\,|r} r^{-1} \sum_{e\,|r/d} g(r/e)eC(n,d) = \sum_{e|r} r^{-1} g(r/e)e \sum_{d|r/e} C(n,d).$$

By Lemma 3

$$\sum_{e\,|r} r^{-1} g(r/e)e \sum_{d\,|r/e} C(n,d) = \sum_{\substack{e|r \\ r/e|n}} g(r/e) = \sum_{d|(n,r)} g(d) = f(n).$$

This completes the proof. □

Examples

Example 1. Ramanujan's sum $C(n,r)$ has the expression $C(n,r) = \sum_{d|(n,r)} d\mu(r/d)$ and is thus easily seen to be even $(\mathrm{mod}\, r)$. According to Theorem 5 its Fourier coefficients are given by $\alpha(d) = 1$ if $d = r$, and $\alpha(d) = 0$ otherwise.

Example 2. *Kronecker's function* ρ_r is defined as

$$\rho_r(n) = \begin{cases} 1 & \text{if } (n,r) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Kronecker's function is even $(\mathrm{mod}\, r)$ and according to Theorem 10 its Fourier coefficients are given by $\alpha(d) = r^{-1}C(r/d,r)$.

Example 3. The function e_0 given as

$$e_0(n) = \begin{cases} 1 & \text{if } n \equiv 0 (\mathrm{mod}\, r), \\ 0 & \text{otherwise} \end{cases}$$

is the identity under the Cauchy product. The function e_0 is even $(\mathrm{mod}\, r)$ and according to Theorem 10 its Fourier coefficients are given by $\alpha(d) = r^{-1}$.

Example 4. The function (n,r) is even $(\mathrm{mod}\, r)$ and according to Theorem 10 its Fourier coefficients are given by $\alpha(d) = r^{-1} \sum_{e|r/d} \phi(r/e)e$.

Example 5. *Nagell's function* $\theta(n,r)$ counts the number of integers a $(\mathrm{mod}\, r)$ such that $(a,r) = (n-a,r) = 1$. Then

$$\theta(n,r) = (\rho_r \circ \rho_r)(n).$$

Thus, using Example 2 and Theorem 6 we obtain

$$\theta(n,r) = r^{-1} \sum_{d|r} C(r/d,r)^2 C(n,d).$$

Example 6. Let $N(n,r,s)$ denote the number of s-vectors $\langle x_1, x_2, \ldots, x_s \rangle$ $(\mathrm{mod}\, r)$ such that

$$x_1 + x_2 + \cdots + x_s \equiv n \pmod{r},$$

where $(x_1,r) = (x_2,r) = \cdots = (x_s,r) = 1$. Then

$$N(n,r,s) = (\underbrace{\rho_r \circ \rho_r \circ \cdots \circ \rho_r}_{s \text{ times}})(n)$$

and therefore $N(n,r,s) = r^{-1} \sum_{d|r} C(r/d,r)^s C(n,d)$. In particular, $N(n,r,2) = \theta(n,r)$.

References

1 E. Cohen, 1955, *A class of arithmetical functions*, Proc. Nat. Acad. Sci. U.S.A., **41**, pp. 939–944.

2 E. Cohen, 1958, *Representations of even functions* $(\mathrm{mod}\, r)$*. I. Arithmetical identities*, Duke Math. J., **25**, pp. 401–421.

3 E. Cohen, 1959, *Representations of even functions* $(\mathrm{mod}\, r)$*. II. Cauchy products*, Duke Math. J., **26**, pp. 165–182.

4 E. Cohen, 1959, *Representations of even functions* $(\mathrm{mod}\, r)$*. III. Special topics*, Duke Math. J., **26**, pp. 491–500.

5 P. J. McCarthy, 1986, *Introduction to Arithmetical Functions*, Universitext, Springer Verlag, New York.

6 R. Sivaramakrishnan, 1989, *Classical Theory of Arithmetic Functions*, in Monographs and Textbooks in Pure and Applied Mathematics, Vol. **126**, Marcel Dekker, Inc., New York.