An elementary linear algebraic approach to even functions (mod \(r\))

An arithmetical function \(f(n)\) is said to be even (mod \(r\)) if \(f(n) = f((n,r))\) for all \(n \in \mathbb{Z}^+\), where \((n,r)\) is the greatest common divisor of \(n\) and \(r\). It is well known that every even function \(f\) (mod \(r\)) possesses a representation \(f(n) = \sum_{d|r} \alpha(d) C(n,d)\) in terms of Ramanujan’s sum \(C(n,d)\). In this paper we interpret two known expressions of \(\alpha\) using elementary linear algebra. We also present the known convolution representation of an even function (mod \(r\)) in a simple way using the Dirichlet convolution.

Introduction
Let \(r\) be a fixed positive integer greater than 1. An arithmetical function \(f(n)\) is said to be even (mod \(r\)) if

\[ f(n) = f((n,r)) \]

for all \(n \in \mathbb{Z}^+\), where \((n,r)\) is the greatest common divisor of \(n\) and \(r\).

The concept of an even function (mod \(r\)) originates with Cohen [1]. Even functions (mod \(r\)) were further studied by Cohen in subsequent papers [2, 3, 4]. General accounts of even functions (mod \(r\)) can be found, for example, in the books by McCarthy [5] and Sivaramakrishnan [6].

It is well known that every even function \(f\) (mod \(r\)) possesses a representation

\[ f(n) = \sum_{d|r} \alpha(d) C(n,d) \]

in terms of Ramanujan’s sum \(C(n,d)\), where

\[ C(n,r) = \sum_{\substack{k \equiv 0 \pmod{r} \atop 1, 2, \ldots, \tau(n)\, \text{unique}}} \exp(2\pi i kn/r). \]

The coefficients \(\alpha(d)\), where \(d|r\), are unique and are referred to as the Fourier coefficients of \(f\). It is well known that

\[ \alpha(d) = r^{-1} \sum_{c|r} f(r/c) C(r/d,c) = (\phi(d))^{-1} \sum_{a \equiv 0 \pmod{r}} f(a) C(a,r), \]

where \(\phi\) is Euler’s totient function. In this paper we interpret these expressions of \(\alpha\) using elementary linear algebra. We also present the known convolution representation of an even function (mod \(r\)) in a simple way using the Dirichlet convolution. Some examples of even functions (mod \(r\)) are also given.

The vector space of even functions (mod \(r\))
Let \(r \geq 2\) be fixed. Let \(E_r\) denote the set of all even functions (mod \(r\)).

**Theorem 1.** The set \(E_r\) forms a complex vector space under the usual sum of functions and the scalar multiplication.

The proof is an easy exercise in elementary linear algebra.

**Theorem 2.** The dimension of the vector space \(E_r\) is \(\tau(r)\), the number of positive divisors of \(r\).

**Proof.** Let \(d_1, d_2, \ldots, d_{\tau(r)}\) be the positive divisors of \(r\) in ascending order. For each divisor \(d_i\), \(i = 1, 2, \ldots, \tau(r)\), define the function \(\rho_i^{(i)}\) as

\[ \rho_i^{(i)}(n) = \begin{cases} 1 & \text{if } (n,r) = d_i, \\ 0 & \text{otherwise}. \end{cases} \]

We prove that \(\{\rho_i^{(i)} : i = 1, 2, \ldots, \tau(r)\}\) is a basis of the vector space \(E_r\). Clearly \(\rho_i^{(i)} \in E_r\) for all \(i = 1, 2, \ldots, \tau(r)\). Every \(f \in E_r\) can be written as a linear combination of the functions \(\rho_i^{(i)}, i = 1, 2, \ldots, \tau(r)\), as

\[ f(n) = f(d_1)\rho_1^{(1)}(n) + f(d_2)\rho_2^{(2)}(n) + \cdots + f(d_{\tau(r)})\rho_{\tau(r)}^{(\tau(r))}(n). \]

Further, it can be verified that the functions \(\rho_i^{(i)}, i = 1, 2, \ldots, \tau(r)\), are linearly independent. This shows that \(\{\rho_i^{(i)} : i = 1, 2, \ldots, \tau(r)\}\) is a basis of the vector space \(E_r\), and thus the dimension of the vector space \(E_r\) is \(\tau(r)\).

\(\square\)

The inner product space of even functions (mod \(r\))
The Dirichlet convolution of arithmetical functions \(f\) and \(g\) is defined by

\[ (f * g)(n) = \sum_{d|r} f(d)g(n/d). \]

**Theorem 3.** Let \(r \geq 2\) be fixed. The vector space \(E_r\) forms a complex inner product space with

\[ (f, g) = \sum_{d|r} \phi(d) f(r/d) \overline{g(r/d)} = (\phi * f \overline{g})(r), \tag{1} \]

where \(\overline{g}(n) = \overline{g(n)}\), the complex conjugate of \(g(n)\).
The proof is an easy exercise in elementary linear algebra.

**Lemma 1.** [5, p. 79] Let \( d_1 \mid r \) and \( d_2 \mid r \). Then
\[
\sum_{e \mid r} C(r/e, d_1)C(r/e, d_2)\phi(e) = \begin{cases} r\phi(d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise}. \end{cases}
\]

**Remark.** If \( d \mid r \), then \( C(n, d) \) is even \((\mod r)\).

**Theorem 4.** The following set is an orthonormal basis of the inner product space \( E_r \):
\[
\{(r\phi(d))^{-\frac{1}{2}}C(\cdot, d) : d \mid r\}. 
\tag{2}
\]

**Proof.** As the dimension of the inner product space \( E_r \) is \( \tau(r) \) and the number of elements in the set \( 2 \) is \( \tau(r) \), it suffices to show the set \( 2 \) is an orthonormal subset of \( E_r \). This follows easily from the above remark and Lemma 1.

**Theorem 5.** An arithmetical function \( f \) is even \((\mod r)\) if and only if it has a representation
\[
f(n) = \sum_{d \mid r} \alpha(d)C(n, d),
\tag{3}
\]
where
\[
\alpha(d) = r^{-1} \sum_{e \mid r} f(r/e)C(r/d, e).
\tag{4}
\]

**Proof.** If \( f \) possesses the representation \( 3 \), then \( f \) is a linear combination of even functions \((\mod r)\). As even functions \((\mod r)\) form a vector space, this linear combination is even \((\mod r)\). Suppose that \( f \) is even \((\mod r)\). Then by Theorem 4
\[
f(n) = \sum_{d \mid r} \langle f, (r\phi(d))^{-\frac{1}{2}}C(\cdot, d) \rangle \langle r\phi(d) \rangle^{-\frac{1}{2}}C(n, d),
\tag{5}
\]
where
\[
\langle f, (r\phi(d))^{-\frac{1}{2}}C(\cdot, d) \rangle = \sum_{e \mid r} \phi(e)f(r/e)\langle r\phi(d) \rangle^{-\frac{1}{2}}C(r/e, d).
\]
It is known [5, p. 93] that \( \phi(e)C(r/e, d) = \phi(d)C(r/d, e) \). Therefore
\[
\langle f, (r\phi(d))^{-\frac{1}{2}}C(\cdot, d) \rangle = r^{-\frac{1}{2}}\phi(d)\sum_{e \mid r} f(r/e)C(r/d, e). 
\tag{6}
\]
Combining formulas \( 5 \) and \( 6 \) gives formula \( 3 \).

**The Cauchy product**

Let \( f \) and \( g \) be even functions \((\mod r)\). Their Cauchy product is defined as
\[
(f \circ g)(n) = \sum_{a+b=n \ (\mod r)} f(a)g(b) = \sum_{a \ (\mod r)} f(a)g(n-a).
\]

**Lemma 2.** [5, p. 76] Let \( d_1, d_2 \mid r \). Then
\[
\sum_{a+b=n \ (\mod r)} C(a, d_1)C(b, d_2) = \begin{cases} rC(n, d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise}. \end{cases}
\]

**Theorem 6.** Let \( f \) and \( g \) be even functions \((\mod r)\) with Fourier coefficients, respectively, \( \alpha_f(d) \) and \( \alpha_g(d) \), \( d \mid r \). Then \( f \circ g \) is even \((\mod r)\) with Fourier coefficients \( r\alpha_f(d)\alpha_g(d), d \mid r \).

**Proof.** By Theorem 5
\[
(f \circ g)(n) = \sum_{a+b=n \ (mod r)} \sum_{e \mid r} \alpha_f(d)C(a, d)\alpha_g(e)C(b, e) = \sum_{a+b=n \ (mod r)} \sum_{e \mid r} \alpha_f(d)\alpha_g(e)C(a, d)C(b, e).
\]
By Lemma 2 \( (f \circ g)(n) = \sum_{e \mid r} r\alpha_f(d)\alpha_g(d)C(n, d) \). Thus \( f \circ g \) is even \((\mod r)\) with Fourier coefficients \( r\alpha_f(d)\alpha_g(d), d \mid r \).

**Theorem 7.** Inner product \( 1 \) can be written as \( \langle f, g \rangle = \langle f \circ g \rangle \).

**Proof.** By Parseval’s identity
\[
\langle f, g \rangle = \sum_{d \mid r} \langle f, (r\phi(d))^{-\frac{1}{2}}C(\cdot, d) \rangle \langle r\phi(d) \rangle^{-\frac{1}{2}}C(n, d).
\]
By equations \( 6 \) and \( 7 \)
\[
\langle f, g \rangle = \sum_{d \mid r} \alpha_f(d)\sum_{e \mid r} \phi(e)f(r/e)\langle r\phi(d) \rangle^{-\frac{1}{2}}C(r/e, d).
\]
This completes the proof of Theorem 7.

**Theorem 8.** The Fourier coefficients of an even function \( f \) \((\mod r)\) have the expression
\[
\alpha(d) = (r\phi(d))^{-1} \sum_{a \ (\mod r)} f(a)C(a, r). 
\tag{7}
\]

**Proof.** According to equation \( 5 \) and Theorem 7
\[
f(n) = \sum_{d \mid r} \sum_{a+b=n \ (mod r)} \sum_{\phi(e)C(r/e, d) = \phi(d)C(r/d, e)} \alpha_f(d)\langle r\phi(d) \rangle^{-\frac{1}{2}}C(b, d)\langle r\phi(d) \rangle^{-\frac{1}{2}}C(n, d)
\]
\[
= \sum_{d \mid r} \langle r\phi(d) \rangle^{-\frac{1}{2}} \sum_{a \ (mod r)} f(a)C(n-a, d)C(n, d).
\]
As \( C(n-a, d) = C(a, d) \), we obtain \( 7 \).

**A convolution representation**

**Theorem 9.** An arithmetical function \( f \) is even \((\mod r)\) if and only if it can be written as
\[
f(n) = \sum_{d \mid (r, n)} g(d),
\tag{8}
\]
where \( g \) is an arithmetical function (which may depend on \( r \)). In this case \( g = f * \mu \), where \( \mu \) is the Möbius function.

**Proof.** Let \( f \) be even \((\mod r)\). As \( \mu \) is the inverse of the constant function \( 1 \) under the Dirichlet convolution, we have
\[
f(n) = f((n, r)) = \sum_{d \mid (n, r)} (f * \mu)(d).
\]
Thus \( f \) possesses (8) with \( g = f \cdot \mu \). Conversely, if (8) holds, then
\[
f(n) = \sum_{d | (n, r)} g(d) = \sum_{d | ((n, r), r)} g(d) = f((n, r)).
\]
Thus \( f \) is even (mod \( r \)).

**Lemma 3.** [5, p. 71] We have
\[
\sum_{d \mid r} C(n, d) = \begin{cases} r & \text{if } r 
mid n, \\
0 & \text{otherwise.}
\end{cases}
\]

**Theorem 10.** The Fourier coefficients of an even function \( f \) (mod \( r \)) have the expression:
\[
\alpha(d) = r^{-1} \sum_{e \mid r/d} (f \cdot \mu)(r/e) e.
\]

**Proof.** Let \( g = f \cdot \mu \). We put the coefficients (9) on the right-hand side of (3) to get
\[
\sum_{d \mid r} \sum_{e \mid r/d} g(r/e) e C(n, d) = \sum_{e \mid r} r^{-1} g(r/e) e \sum_{d \mid r/e} C(n, d).
\]
By Lemma 3
\[
\sum_{e \mid r} \sum_{r/e} g(r/e) e C(n, d) = \sum_{r/e} g(r/e) = \sum_{d \mid (n, r)} g(d) = f(n).
\]
This completes the proof.

**Examples**

**Example 1.** Ramanujan’s sum \( C(n, r) \) has the expression \( C(n, r) = \sum_{d \mid (n, r)} d \mu(r/d) \) and is thus easily seen to be even (mod \( r \)). According to Theorem 5 its Fourier coefficients are given by \( \alpha(d) = 1 \) if \( d = r \), and \( \alpha(d) = 0 \) otherwise.

**Example 2.** Kronecker’s function \( \rho_r \) is defined as
\[
\rho_r(n) = \begin{cases} 1 & \text{if } (n, r) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]
Kronecker’s function is even (mod \( r \)) and according to Theorem 10 its Fourier coefficients are given by \( \alpha(d) = r^{-1} C(r/d, r) \).

**Example 3.** The function \( e_0 \) given as
\[
e_0(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{r}, \\
0 & \text{otherwise.}
\end{cases}
\]
is the identity under the Cauchy product. The function \( e_0 \) is even (mod \( r \)) and according to Theorem 10 its Fourier coefficients are given by \( \alpha(d) = r^{-1} \).

**Example 4.** The function \( (n, r) \) is even (mod \( r \)) and according to Theorem 10 its Fourier coefficients are given by \( \alpha(d) = r^{-1} \).

**Example 5.** Nagell’s function \( \theta(n, r) \) counts the number of integers \( a \) (mod \( r \)) such that \((a, r) = (n - a, r) = 1\). Then
\[
\theta(n, r) = (\rho_r \circ \rho_r)(n).
\]
Thus, using Example 2 and Theorem 6 we obtain
\[
\theta(n, r) = r^{-1} \sum_{d \mid r} C(r/d, r)^2 C(n, d).
\]

**Example 6.** Let \( N(n, r, s) \) denote the number of \( s \)-vectors \((x_1, x_2, \ldots, x_s) \) (mod \( r \)) such that
\[
x_1 + x_2 + \cdots + x_s \equiv n \pmod{r},
\]
where \((x_1, r) = (x_2, r) = \cdots = (x_s, r) = 1\). Then
\[
N(n, r, s) = (\rho_r \circ \rho_r \circ \cdots \circ \rho_r)(n)
\]
s times and therefore
\[
N(n, r, s) = r^{-1} \sum_{d \mid r} C(r/d, r)^s C(n, d).
\]
In particular, \( N(n, r, 2) = \theta(n, r) \).

**References**