Problemen

| Problem Section

Problem 10 (Open problem, by Frits Beukers)
Let a, b, c be integers such that the symmetric matrix

$$\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$$

has three integer eigenvalues. Prove, or give a counter example to, the following statement: either $abc = 0$ or $(a^2 - b^2)(a^2 - c^2)(b^2 - c^2) = 0$.

The following problems 11–14, proposed by Hendrik Lenstra, are related to the *abc-conjecture*. An *abc-triple* is a triple of pairwise coprime positive integers a, b, c with $a + b = c$ for which the product r of the distinct prime numbers dividing abc satisfies $r < c$. For example, the equality $32 + 49 = 81$ gives rise to the *abc*-triple 32, 49, 81, with $r = 2 \cdot 7 \cdot 3 = 42 < 81 = c$. The *abc-conjecture* states that $(\log c)/\log r$ tends to 1 as the limit is taken over all *abc*-triples. It is apparently not even known whether there is an infinite set of *abc*-triples for which $(\log c)/\log r$ tends to 1.

Problem 11
Prove that there are infinitely many *abc*-triples.

Problem 12
Prove that there are infinitely many *abc*-triples for which a is equal to a given positive integer.

Problem 13
Let m be a positive integer. Prove that there is an *abc*-triple with the property that any odd prime number dividing abc exceeds m.

Problem 14
Let n be a positive integer. Prove that there exist n different *abc*-triples with the same value of c.

Solutions to volume 1, number 2 (June 2000)

Problem 4
Eleven journalists have their own bit of slander. They possess special telephones that allow 3 men to communicate with each other. How many calls are needed to inform everyone of everyone else's information?

This is a variation on the Telephone Problem. The Telephone Problem is exercise 58 *on page* 32 *of Bollobás, Modern Graph Theory.*

Solution by Aad van de Wetering (CBS, Voorburg): 8 calls. The journalists are A, B, C, et cetera. The order of the telephone calls is given in a matrix.

A	B	C	D	E	F	G	H	I	J	K
1	1	1	2	2	2	3	3	3		
								4	4	4
5	6	7	5	6	7			5	6	7
						8	8	8		

One verifies that it takes at least 5 calls before one of the journalists gets all the information. After 4 calls all 11 journalists are waiting for more information. After 5 calls 8 journalists are waiting, et cetera. So it takes 8 calls.

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

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Oplossingen | Solutions

Problem 5
Construct a countable, compact subset of the plane, not contained in a line, that intersects no line in 2 points.

For any finite subset F of the plane there exists a line that intersects F in two points. This is Sylvester's problem. A line that intersects F in two points is called a Gallai line. Countable and compact is as close to finite as an infinite set can get. There exist subsets V of the plane for which every line is a Gallai line. It is an old problem in set theory whether V can be Borel. Khalid Bouhjar (VU Amsterdam) has some results on his homepage http://www.cs.vu.nl/~kbouhjar

G.A. Kootstra (Broek op Langedijk) gives the following example, for some constant $r \notin \mathbf{Z}$:

$$V = \left\{ \left(-\frac{1}{2n+r}, \frac{1}{2n+r}\right) \cup \left(\frac{1}{2n+r}, \frac{1}{2n+r}\right) \cup \left(0, \frac{1}{n+r}\right) \cup (0,0) \;:\; n \in \mathbf{Z} \right\}.$$

Every line intersects V in 0, 1, 3 or infinitely many points. Kootstra remarks that this example was given by Peter Borwein, Sylvester's problem and Motzkin's theorem for countable and compact sets, Proc. Amer. Math. Soc. 90 (1984), no. 4, 580-584. Borwein states that the example is 'from the folk literature' and that 'it would be interesting to know if any others exist'.

Problem 6
Define a_n, for $n \geq 0$ by $a_0 = 3, a_1 = 0, a_2 = 2$ and $a_{n+3} = a_n + a_{n+1}$ for all $n \geq 0$. Show that $p|a_p$ for every prime number p.

Solutions by Wim Luxemburg (Pasadena) and Frits Beukers (Utrecht). Let θ be a zero of $x^3 - x - 1$. Then $\theta^{n+3} = \theta^{n+1} + \theta^n$ for any integer n and we see that θ^n, $n = 0, 1, 2, 3, \ldots$ is a solution of the recurrence. Let $\theta_1, \theta_2, \theta_3$ be the three zeros of $x^3 - x - 1$. Then, by linearity,

$$\theta_1^n + \theta_2^n + \theta_3^n, \; n = 0, 1, 2, \ldots$$

again satisfies the recurrence. Moreover, one easily checks that the first three terms equal $3, 0, 2$. Conclusion,

$$a_n = \theta_1^n + \theta_2^n + \theta_3^n$$

for all n. Let now p be a prime. Beukers now uses the identity

$$X^p + Y^p + Z^p = (X + Y + Z)^p$$

in the field $\mathbf{F_p}(\theta_\mathbf{i})$ to derive

$$a_p = \theta_1^p + \theta_2^p + \theta_3^p \equiv (\theta_1 + \theta_2 + \theta_3)^p \equiv a_1^p \equiv 0 \pmod{p}.$$

Hence p divides a_p. Luxemburg uses a result of Schönemann (Crelle 32, (1846)) to obtain this result. Beukers remarks that the converse statement, n divides $a_n \Rightarrow n$ is prime, is known to be false, but that the first counter example is very large.

Solutions to some problems of yore

The Problem Section of the previous series of Nieuw Archief ended rather abruptly, as the solutions to Problems 950-979 had not yet been published. We apologize for letting the contributors to the Problem Section wait for so long. There are several relentless problem solvers, most notably, with descending number of solutions, A.A. Jagers (20), H.J. Seiffert (14), G.W. Veltkamp (10), R.A. Kortram (10), J.H. van Geldrop (8), D. Constales (6), R.H. Jeurissen (6), F.J.H. Barning (5), J. Boersma (5), and K.W. Lau (5). We thank the originators of these problems for their contribution, but above all thanks are due to Professor M.L.J. Hautus who has taken care of the Problem Section since 1980 and sustained an average of ten problems per issue. Below are the solutions to the problems 950–960. Solutions to the remaining problems 961–979 will be published in the next issue of Nieuw Archief.

Oplossingen | Solutions

Problem 950 (H. Alzer)
Let $(a_i), (b_i), (p_i)$ be positive real numbers for $i = 1, \ldots, n$ satisfying $\sum_{i=1}^n p_i = 1$. Prove the following statements:

- If (a_i) and (b_i) are both non-decreasing or both non-increasing. Then

$$\sum_{i=1}^n p_i a_i^2 b_i \sum_{i=1}^n p_i b_i + \sum_{i=1}^n p_i a_i b_i^2 \sum_{i=1}^n p_i a_i \le \sum_{i=1}^n p_i a_i^2 b_i^2 + \left(\sum_{i=1}^n p_i a_i b_i\right)^2. \quad (1)$$

- If (a_i) and (b_i) are non-decreasing and (b_i) is non-increasing, then the reverse inequality is valid.

Solutions by H.J. Seiffert, R.H. Jeurissen, F.J.H. Baring, R.A. Kortram, G.W. Veltkamp. H.J. Seiffert remarks that (1) is a special case of an inequality due to Bencze. The solutions are similar. Below is R.H. Jeurissen's solution. Using automatic index-summation, the left-hand side L of (1) equals $p^i p^j a_i b_i (a_i b_j + b_i a_j)$, while the right-hand side R equals $p^i p^j a_i^2 b_i^2 + p^i p^j a_i b_i a_j b_j$. Now $R - L = p^i p^j a_i b_i (a_i - a_j)(b_i - b_j)$. Check the sign of the product $(a_i - a_j)(b_i - b_j)$.

Problem 951 (W. Bencze)
Let D denote the unit circle in the plane and let $A_1, \ldots, A_n$ be points on D. Prove that $\max \prod_{k=1}^n PA_k \ge 2$, where PA_k denotes the distance between P and A_k. The equality holds if and only if the A_i form a regular polygon.

Solutions by D. Constales, A.N. 't Woord, A.A. Jagers, R.A. Kortram. H.J. Seiffert remarks that this is Problem Q852 of Math. Mag. 69 vol 3, 1996, posed by the same author. All solutions involve complex variables and depend upon the equality

$$\frac{1}{n}\Big(f(z) + f(\zeta z) + f(\zeta^2 z) + \ldots + f(\zeta^{n-1} z)\Big) = z^n + f(0)$$

if f is a monic polynomial of degree n and ζ is a primitive n-th root of unity.

Problem 952 (P.J. de Doelder †)
Show that $\sum_{n=0}^\infty \frac{(1)_n}{(3/2)_n}\left[\psi(n+3/2) - \psi(n+1)\right] = 2\log 2$, where $(a)_n = \Gamma(a+n)/\Gamma(a)$ and $\psi(a) = \Gamma'(a)/\Gamma(a)$ and Γ denotes the gamma function.

Solutions by J. Boersma, A.A. Jagers, H.J. Seiffert, D. Constales. Solution by Constales. Let $x > -1$ and $n > 0$. Express the Beta function in terms of the Gamma function

$$\frac{\Gamma(n+x+1)\Gamma(1/2)}{\Gamma(n+x+3/2)} = \int_0^1 \frac{u^{n+x}}{\sqrt{1-u}} du.$$

Let $|\alpha| < 1$ and multiply by α^n and sum over n to obtain

$$\sum_{n=0}^\infty \frac{\Gamma(n+x+1)\Gamma(1/2)}{\Gamma(n+x+3/2)}\alpha^n = \int_0^1 \frac{u^x}{(1-\alpha u)(\sqrt{1-u})} du.$$

Take the derivative with respect to x at $x = 0$

$$\sum_{n=0}^\infty \left(\frac{\Gamma'(n+1)\Gamma(1/2)}{\Gamma(n+3/2)} - \frac{\Gamma(n+1)\Gamma(1/2)\Gamma(n+3/2)}{\Gamma(n+3/2)^2}\right)\alpha^n = \int_0^1 \frac{\log u}{(1-\alpha u)(\sqrt{1-u})} du.$$

The left-hand side is -2 times the required sum. The right-hand side is an elementary integral and taking the limit $u \to 0$ gives the required solution.

Problem 953 (P. de Groen)
Prove that

$$f(t) = \int_0^t \int_s^\infty \exp(s^2 - u^2) du ds = \frac{1}{2}\log t + C + O\Big(\frac{1}{t^2}\Big),$$

and determine the constant C.

Solutions by D. Constales, A.A. Jagers, J. Boersma, H.J. Seiffert, G.W. Veltkamp, Kee-Wai Lau. The evaluation of the constant C requires some equalities between special functions involving Euler's constant γ, with the result that $C = \frac{\log 2}{2} + \frac{\gamma}{2}$. A.A. Jagers gives the full asymptotic expansion of f! Here is H.J. Seiffert's elementary solution. Some calculation gives that f satisfies the differential equation $f' = \frac{1}{2t} + \frac{f''}{2t}$. By integration we find that

$$f = \frac{1}{2}\log t + C + \int_t^\infty \frac{f''(s)}{2s}\,ds = \frac{1}{2}\log t + C + \int_t^\infty \int_s^\infty \exp(s^2 - u^2)\,duds.$$

Substitute $u = v + t$ in this double integral to get

$$\int_t^\infty \int_0^\infty \frac{v}{s}\exp(-2sv - v^2)\,dvds \le \int_t^\infty \frac{1}{s}\int_0^\infty v\exp(-2sv)\,dvds = \frac{1}{8t^2}.$$

Problem 954 (M.L.J. Hautus)

Let R be an integral domain with identity 1 and let $n > 1$ be an integer. Denote the set of $n \times n$ matrices with entries in R by $R^{n\times n}$ and define the matrix $N \in R^{n\times n}$ by

$$\begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 0 & 1 & 0 \\ \vdots & & & 0 & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 & 0 \end{bmatrix}.$$

Show that for a polynomial $p(z) \in R[z]$ the equation $p(X) = N$ has a solution in $R^{n\times n}$ if and only of there exists a $\lambda \in R$ such that $p(\lambda) = 0$ and $p'(\lambda)$ is a unit in R.

Solutions by D. Constales, A.A. Jagers, G.W. Veltkamp. Below is the solution by D. Constales. First observe that N is nilpotent $N^n = 0$. Define the polynomial $q(z) = p(z+\lambda)$ and denote $p'(\lambda) = q_1$. Then $q(z) = q_1 z + z^2 r(z)$. The problem is solved once we find a matrix Y such that $p(Y) = N$, which is equivalent to a fixed point of $Y \to q_1^{-1}(N - Y^2 r(Y))$. Define $Y_k \in R^{n\times n}$ recursively by $Y_0 = 0$ and $Y_{k+1} = q_1^{-1}(N - Y_k^2 r(Y_k))$. Then each Y_k can be expressed as a polynomial $s_k(N)$ with zero constant term. One can check that the k-th coefficient of s_k stabilizes and by the nilpotency of N the sequence Y_k stabilizes for $k \ge n$, which gives the required fixed-point.

To prove the necessity of the condition, observe that if $p(X) = N$ then X commutes with N. Let e_i denote the standard basis. Then Xe_1 belongs to the kernel of N, so $Xe_1 = \lambda e_1$ for some $\lambda \in R$. Check that $p(\lambda) = 0$. We now show that $p'(\lambda)$ is a unit in R. Expand $p(z)$ around λ to get

$$N = p(X) = p(\lambda) + p'(\lambda)(X-\lambda) + O((X-\lambda)^2) = p'(\lambda)(X-\lambda) + O((X-\lambda)^2).$$

Since $Ne_2 = e_1$ and since $(X-\lambda)e_1 = 0$ it follows that $(X-\lambda)e_2$ belongs to the kernel of N. So $(X-\lambda)^2 e_2 = 0$. It follows that $e_1 = Ne_2 = p(X)e_2 = p'(\lambda)\mu e_1$, so μ is a unit in R.

Problem 955 (J. van de Lune)

Let $Q[x]$ denote the set of all polynomials with rational coefficients, which is a vector space over Q. Define an operator on this vector space by

$$Tf(x) = xf'(x) - f(x+1) + f(1).$$

Find all eigenvalues of T and describe an efficient procedure to compute the eigenpolynomials.

Solutions by D. Constales, R.H. Jeurissen, A.A. Jagers, B. Burghgraef, J. Boersma. Below is the solution by R.H. Jeurissen. We restrict ourselves to monic polynomials and easily find that 1 and $x + a$ ($a \in Q$) and $x^2 - 2x$ are the only eigenpolynomials of degree ≤ 2. Their eigenvalues are 0, 0 and 1, respectively. Comparing the highest coefficients of $\lambda f(x)$ and $Tf(x)$ and their values for $x = 0$ we find that an eigenpolynomial of degree $n \geq 2$ has eigenvalue $n - 1$ and constant term 0. So all eigenvalues are natural numbers and the eigenvalue 0 has no other eigenpolynomials then those in the subspace spanned by 1 and x. Let P_n denote the set of monic polynomials of degree n and let $P_{n,0}$ denote its subset of polynomials with constant term 0. For $n \geq 3$ define the mappings $\phi_n \colon P_{n,0} \to P_{n-1}$ and $\psi_n \colon P_{n-1} \to P_{n,0}$ by

$$\phi_n : f(x) \to \frac{f'(x)}{n} + \frac{f'(1)}{n(n-2)} \quad \text{and} \quad \psi_n \colon g(x) \to \int_0^x g(t)dt - \frac{ng(1)x}{n-1}.$$

A simple verification shows that
- $\psi_n \circ \phi_n = id$;
- If $f \in P_{n+1}$ is an eigenpolynomial (for n) then so is $\phi_n(f)$ (for $n-1$);
- If $g \in P_n$ is an eigenpolynomial (for $n-1$) then so is $\psi_n(g)$ (for n).

It then follows that all eigenvalues can be found from $x^2 - 2x$ by repeated application of the ψ_n.

Problems 956, 957 and 958
These poblems did not receive a response from the readers of NAW and are skipped.

Problem 959 (B.M.M. de Weger)
Prove that for $\alpha \downarrow 0$.

$$\int_0^1 \frac{dx}{\sqrt{x(x+\alpha)(1-x)}} = -\log\alpha + 4\log 2 + \frac{\alpha}{4}\log\alpha + (\frac{1}{2} - \log 2)\alpha + O(\alpha^2 \log\alpha).$$

Solutions by S. Rienstra, A.A. Jagers, D. Constales, H.J. Seiffert, J. Boersma, G.W. Veltkamp. Below is the solution by D. Constales. The substitution $x = \cos^2\theta$ transforms the integral directly to

$$\frac{2}{\sqrt{(1+a)}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1/(1+a))\sin^2\theta}} = \frac{2K(1/\sqrt{1+a})}{\sqrt{1+a}}.$$

in terms of the complete elliptic integral K. The required asymptotic expansion follows at once from that of $K(k)$ near $k = 1$. This can be found in section 8.12 and exercise 8.13 in Lawden, D.F., *Elliptic functions and Applications*, Springer Verlag, 1989.

Problem 960 (H. Alzer)
Let $\alpha > 1$ be a real number. Find the smallest number c_α such that for all non-negative sequences (a_k) with $\sum_{k=1}^{\infty}(ka_k)^\alpha < \infty$ we have $(\sum_{k=1}^{\infty} a_k)^\alpha \leq c_\alpha \sum_{k=1}^{\infty}(ka_k)^\alpha$.

Solutions by A.A. Jagers, R.A. Kortram, J. Boersma, H.J. Seiffert, J.H. van Geldrop. Below is the solution by A.A. Jagers. The notation suppresses that the sums run over the index k from 1 to ∞. The space of α-summable positive sequences is denoted $l_\alpha^+ = \{(x_1, x_2, \ldots) \mid x_k \geq 0,\ \sum x_k^\alpha < \infty\}$. Then

$$c_\alpha = \sup_{(k\alpha_k)\in l_\alpha^+} \frac{(\sum a_k)^\alpha}{\sum(k\alpha_k)^\alpha} = \left[\sup_{(k\alpha_k)\in l_\alpha^+} \frac{\sum ka_k k^{-1}}{\{\sum(k\alpha_k)^\alpha\}^{1/\alpha}}\right]^\alpha = \left[\sum k^{-\beta}\right]^{\alpha/\beta},$$

by Hölder's inequality, where β is given by $\beta^{-1} + \alpha^{-1} = 1$. In other words $c_\alpha = \{\zeta(a/(a-1))\}^{\alpha-1}$ where $\zeta(s)$ denotes the Riemann zeta function.