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# How to spot

**Most optimization problems with continuous variables do not allow analytical solutions and have to be solved numerically. But the remaining small minority contains a great many gems of considerable interest. Among these are for example problems of finding optimal numerical methods to solve optimization problems. The core of their analysis is the development of methods for isolating the optima. Here mathematical rigour is not essential: the verification that a candidate-optimum is a true optimum is usually not difficult. Therefore the name of the game is how to *spot* the candidate optima.**

In this paper we give an intuitive introduction to the main ideas underlying these methods and present a number of applications of their use. This account leads up to the well-known analytical unification of these methods by Tikhomirov. This unification is in the spirit of Lagrange's celebrated multiplier rule. Finally we outline a new, geometric unification which is in the spirit of Fermat's method for spotting optima: put the derivative equal to zero. This unification is simpler and there is reason for hope that it can be used to solve new types of problems. From the unification in the style of Fermat one can derive the unification in the style of Lagrange, and so in particular Pontrijagin's Maximum Principle from Optimal Control.

### Weierstrass' theorem

The existence of solutions of optimization problems is taken care of by the theorem of Weierstrass. One variant of this result is that a continuous function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  which is coercive ( $\Leftrightarrow f(x) \rightarrow \infty$

if  $\|x\|_2 \rightarrow \infty$ ) has a minimum. In the applications we will make repeated use of the theorem of Weierstrass. Let us give a first application.

**Fundamental theorem of algebra.** *A polynomial  $p(z)$  of degree  $n \geq 1$  with complex coefficients has a complex root.*

**Proof.** For each complex number  $z_0$  which is not a root of  $p(z)$  we can write the polynomial  $q(z) = p(z_0 + z)$  as  $q(z) = a_0 + a_k z^k + \dots + a_n z^n$  with  $k \geq 1$  and  $a_0 a_k \neq 0$ . Then — writing  $\beta = \arg(\bar{a}_0 a_k)$  and using that  $|w|^2 = w\bar{w}$  for all complex numbers  $w$  — one has for  $t \in (0, \infty)$  and  $\theta \in \mathbf{R}$  that  $|q(te^{i\theta})|^2$  equals  $|a_0|^2 + 2|a_0||a_k|t^k \cos(k\theta + \beta) + O(t^{k+1})$  (for  $t \downarrow 0$ ). It follows from this expression that  $|q(z)|$  is not minimal in  $z = 0$ , because it is possible to choose  $\theta$  such that  $\cos(k\theta + \beta) < 0$ . As  $z_0$  is an arbitrary complex number with  $p(z_0) \neq 0$ , this proves that if the function  $|p(z)|$  has a minimum, then it must be in a root of  $p(z)$ . Well,  $|p(z)|$  has certainly a minimum; this follows from Weierstrass' theorem as it is a continuous, coercive function on  $\mathbf{C} \simeq \mathbf{R}^2$  (write  $z = x + iy$ ).  $\square$

### Fermat's theorem

The most popular method to spot candidate solutions 'put the derivative equal to zero' was first mentioned by Kepler in his book on the art of making wine barrels [1]. The first proof — for polynomial functions of one variable — was given by Fermat.

# an optimum

The ideal of this method is achieved for a differentiable strictly convex coercive function  $f$  of one variable, as in figure 1. We will call minimizing such a function an ‘ideal’ problem. Then  $f$  has precisely one minimum, the unique root of  $f'(x) = 0$ .

The method of Fermat and also the concept ‘ideal problem’ can be generalized easily to functions of several variables and even to functions on normed vectorspaces.

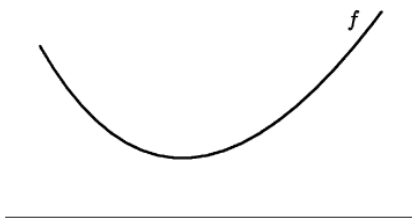


Figure 1 An ideal problem for the method of Fermat

### Do three lines in space have a unique waist?

Many years ago dr. John Tyrrell challenged the PhD students of King’s College London with the following puzzle.

Show that three lines in space in sufficiently general position have a unique waist. This can be visualized as follows. An elastic band is stretched around three lines of iron wire in space which have a fixed position. By elasticity it will slip to a position where its total circumference is minimal. The challenge is to show that this final position does not depend on the initial position of the elastic band; it depends only on the position of the three lines of

iron wire. A precise formalization of the problem is suggested by figure 2: let  $l_1, l_2$  and  $l_3$  be three lines in three-dimensional space, pairwise disjoint and not all mutually parallel. Consider the following minimization problem, where  $\| \cdot \|_2$  is the euclidean norm:

$$(P) = (P_{l_1, l_2, l_3})$$

$$f(p_1, p_2, p_3) = \|p_1 - p_2\|_2 + \|p_2 - p_3\|_2 + \|p_3 - p_1\|_2 \rightarrow \min$$

subject to  $p_i \in l_i$  ( $i = 1, 2, 3$ ).

The problem is to show that  $(P)$  has a unique solution.

Dr. Tyrrell told us that to the best of his knowledge the solution of this simple-looking problem was not known. The words of John carried great weight: he was an expert in all sorts of puzzles. We tried to solve it, for example by eliminating the constraints, applying Fermat’s theorem and carrying out all sorts of algebraic manipulations on the resulting equations. Nothing worked.

Recently I came again across the problem. This time it offered no resistance: the following elementary insight into optimization problems allows a straightforward solution of this puzzle. A successful analysis usually depends on the exploitation of the smoothness and (strict) convexity of the data of the problem at hand. Here the objective function  $f$  turns out to be differentiable, strictly convex and coercive on the affine space of feasible triplets  $(p_1, p_2, p_3)$ . Therefore the problem has a unique solution and this is characterized by ‘the derivative of  $f$  is equal to zero’. That is, we have again an ‘ideal problem’, in the sense given above. Let us

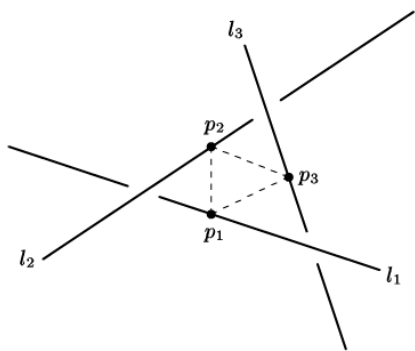


Figure 2 An elastic band stretched around three wires of iron

verify this. It is obvious that  $f$  is differentiable and coercive on the affine space of feasible triplets  $(p_1, p_2, p_3)$ . It remains to check the strict convexity. To do this we use that the euclidean norm  $\|\cdot\|_2$  is a convex function and that its restriction to each line *not* through the origin is strictly convex. This follows for example from the observation that the graph of  $\|\cdot\|_2$  is the ‘icecream cone’, which is shown in figure 3.

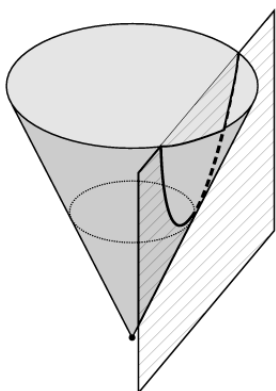


Figure 3 The euclidean norm is ‘almost’ strictly convex

To prove the strict convexity of  $f$  it suffices to take an arbitrary line  $m$  in the affine space of feasible triplets  $(p_1, p_2, p_3)$  and to prove that the restriction of  $f$  to  $m$  is strictly convex. Now  $f$  is defined as the sum of three terms, so it suffices to prove that each of them is convex and that at least one of them is strictly convex. The convexity of these three terms, as functions of a feasible triplet  $(p_1, p_2, p_3)$  follows immediately from the convexity of the euclidean norm  $\|\cdot\|_2$ . Now we take a parametric description  $(p_1(t), p_2(t), p_3(t))$  of the line  $m$  where the  $p_i(t)$  ( $i = 1, 2, 3$ ) are affine functions of one real variable  $t$ . Then not all of the three difference functions  $p_1(t) - p_2(t)$ ,  $p_2(t) - p_3(t)$ ,  $p_3(t) - p_1(t)$  can be constant, as the lines  $l_1, l_2, l_3$  are not all mutually parallel. Without restricting the generality of the argument we assume that  $p_1(t) - p_2(t)$  is not constant. Then  $p_1(t) - p_2(t)$  is a parametric description of a line in  $\mathbf{R}^3$  not through the origin — as  $l_1$  and  $l_2$  have no common points. Therefore  $\|p_1(t) - p_2(t)\|_2$  is a strictly convex function of  $t$ , as desired.

It is possible to give a simple geometric description of the condition that ‘the derivative of  $f$  is equal to zero’. Consider three lines  $l_1, l_2, l_3$  in three-dimensional space satisfying the two assumptions above and let  $p_1, p_2, p_3$  be three distinct points in three-

dimensional space. Let  $b$  be the intersection of the bisectrices of the triangle with vertices  $p_1, p_2, p_3$ . Then the triplet  $(p_1, p_2, p_3)$  is the — unique — solution of the problem  $(P_{l_1, l_2, l_3})$  precisely if  $p_i$  is the orthogonal projection of the point  $b$  on the line  $l_i$  (for  $i = 1, 2, 3$ ).

Finally let us discuss the two assumptions on the lines  $l_1, l_2, l_3$  which we have made. The second assumption is made out of necessity: three parallel lines have clearly no unique waist. The first one is made for the sake of convenience: otherwise  $f$  is not differentiable everywhere. However the method above can be pushed to show that without this assumption one has also uniqueness in all cases except the following one: two of the lines  $l_1, l_2, l_3$  are parallel and the third one intersects both of them.

**Interior point methods**

In 1984 when Karmarkar published his epoch-making paper [2], interior point methods for solving linear programming (LP) problems seemed rather mysterious. Now the basic idea can be explained in a relatively straightforward way. For the intricacies of the method and its implementations we refer for example to [3]. For each linear subspace  $\tilde{P}$  of  $\mathbf{R}^n$  and each vector  $s$  in  $\mathbf{R}^n$  we let  $P = \tilde{P} + s$  and  $D = \tilde{D} + s$ , where  $\tilde{D}$  is the orthogonal complement of  $\tilde{P}$ . We consider the problem:

$$(Q) \quad \begin{aligned} &\text{find two nonnegative vectors } p \in P \\ &\text{and } d \in D \text{ which are orthogonal.} \end{aligned}$$

For practical purposes it usually suffices to find for a given  $\epsilon > 0$  an  $\epsilon$ -solution of (Q), that is two non-negative vectors  $p \in P$  and  $d \in D$  with inner product  $\langle p, d \rangle$  smaller than  $\epsilon$ .

As a first illustration let  $n = 2$  and let  $P$  and  $D$  be two orthogonal lines in the plane  $\mathbf{R}^2$  as in figure 4. Assume that both lines contain positive vectors and do not contain the origin. Then the problem (Q) has a unique solution  $(\hat{p}, \hat{d})$ : one glance at the picture suffices to spot it.

The next case is already slightly more interesting: take  $n = 3$ , choose  $P$  to be a line in  $\mathbf{R}^3$  and  $D$  a plane in  $\mathbf{R}^3$  orthogonal to the line  $P$ . Then the problem (Q) asks to find a point  $p$  in  $P$  and a point  $d$  in  $D$ , both in the first orthant such that the vectors  $p$  and  $d$  are orthogonal. Now we are going to give a geometrical

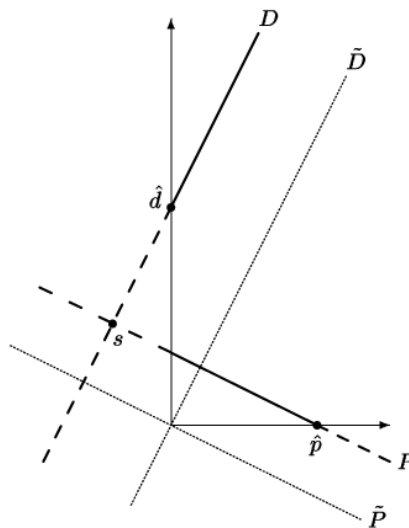


Figure 4 The LCP-problem: primal and dual LP-problems in one picture

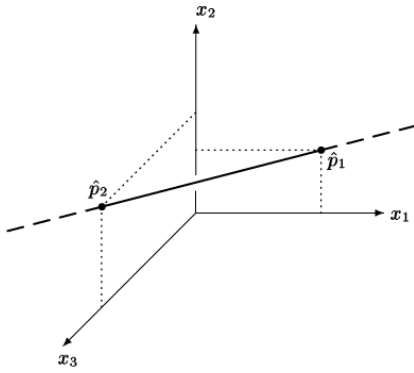


Figure 5 The LCP-problem: primal and dual LP-problems in one picture (in space)

description of the unique solution of this problem in the following special case. The line  $P$  intersects the  $x_1$ - $x_2$ -plane (respectively the  $x_2$ - $x_3$ -plane) in a point  $\hat{p}_1$  (respectively  $\hat{p}_2$ ) which lies in the interior of the first quadrant of this plane, as is shown in figure 5.

Moreover we assume that the point  $\hat{d}$  of intersection of  $D$  with the union of the positive  $x_1$ -axis and the positive  $x_3$ -axis is not equal to the origin. Then the problem  $(Q)$  has a unique solution. It is  $(\hat{p}_2, \hat{d})$  if  $\hat{d}$  lies on the  $x_1$ -axis: then  $\hat{p}_2$  and  $\hat{d}$  are orthogonal. It is  $(\hat{p}_1, \hat{d})$  if  $\hat{d}$  lies on the  $x_3$ -axis: then  $\hat{p}_1$  and  $\hat{d}$  are orthogonal.

If  $n$  is large then the combinatorics of the situation is sufficiently rich to make the problem  $(Q)$  really interesting. A problem  $(Q)$  is called a *linear complementarity problem* (LCP). This terminology is motivated by the observation that the orthogonality condition for two nonnegative vectors  $p$  and  $d$  is equivalent to the complementarity conditions  $p_i d_i = 0 \ (\forall i)$ . Now we will relate the problem  $(Q)$  to the following pair of primal-dual LP-problems

$$\begin{aligned} (P) \quad & f(p) = \langle s, p \rangle \rightarrow \min \quad \text{subject to } p \in P \text{ and } p \geq 0, \\ (D) \quad & g(d) = \langle s, s - d \rangle \rightarrow \max \quad \text{subject to } d \in D \text{ and } d \geq 0. \end{aligned}$$

Let  $\varepsilon > 0$  be given. We call a feasible vector  $p$  for  $(P)$  an  $\varepsilon$ -solution of  $(P)$  if  $f(p)$ -value $(P) < \varepsilon$  where value $(P)$  is defined to be the infimum of all values taken by  $f$  on feasible vectors for  $(P)$ . In a similar way one defines the concept  $\varepsilon$ -solution for the maximization problem  $(D)$ . The promised relation of  $(Q)$  with  $(P)$  and  $(D)$  is as follows: if  $(\hat{p}, \hat{d})$  is an  $\varepsilon$ -solution of  $(Q)$  then  $\hat{p}$  is an  $\varepsilon$ -solution of  $(P)$  and  $\hat{d}$  is an  $\varepsilon$ -solution of  $(D)$ .

Let us verify this. Let  $(\hat{p}, \hat{d})$  be an  $\varepsilon$ -solution of  $(Q)$ . Take arbitrary feasible vectors  $p$  of  $(P)$  and  $d$  of  $(D)$ . Then the difference vector  $p - s$  (respectively  $d - s$ ) lies in  $\tilde{P}$  (respectively in  $\tilde{D}$ ). Therefore  $\langle p - s, d - s \rangle = 0$ . Rewriting this gives  $\langle s, p \rangle - \langle s, s - d \rangle = \langle p, d \rangle$ . This is  $\geq 0$  and moreover it is  $< \varepsilon$  if  $p = \hat{p}$  and  $d = \hat{d}$ . As  $p$  (respectively  $d$ ) is an *arbitrary* feasible vector of the minimization problem  $(P)$  (respectively the maximization problem  $(D)$ ), this implies that  $\hat{p}$  is an  $\varepsilon$ -solution of  $(P)$  and that  $\hat{d}$  is an  $\varepsilon$ -solution of  $(D)$ .

Now we turn to the problem of finding an  $\varepsilon$ -solution of the LCP-problem  $(Q)$  and so of the primal-dual pair of LP-problems  $(P)$  and  $(D)$ . To this end we introduce an auxiliary problem  $(Q_x)$  for each positive vector  $x \in \mathbf{R}^n$ . To define this and for other purposes we introduce a notation for the extension of operations on numbers to pointwise operations on vectors:

$$v \cdot w = (v_1 w_1, \dots, v_n w_n) \quad \forall v, w \in \mathbf{R}^n \text{ ('the Hadamard product')},$$

$$\begin{aligned} \ln v &= (\ln v_1, \dots, \ln v_n) \quad \text{for all positive vectors } v \in \mathbf{R}^n, \\ v^r &= (v_1^r, \dots, v_n^r) \quad \text{for all positive vectors } v \in \mathbf{R}^n \text{ and all } r \in \mathbf{R}. \end{aligned}$$

These notations allow a convenient way of defining  $(Q_x)$ :

$$\begin{aligned} (Q_x) \quad & h_x(p, d) = \langle p, d \rangle - x \cdot \ln(p \cdot d) \rightarrow \min \\ & \text{subject to } p \in P, p > 0, d \in D, d > 0. \end{aligned}$$

This is again an 'ideal problem', provided that feasible pairs  $(p, d)$  exist. Indeed one can readily check that the objective function of  $(Q_x)$  is a differentiable, strictly convex, coercive function. Therefore  $(Q_x)$  has a unique solution, say  $(p(x), d(x))$  and this solution can be characterized by the condition 'the derivative of the objective function of  $(Q_x)$  is zero'. The explicit form of this condition is  $p \cdot d = x$ .

Let us verify this. For each positive  $x \in \mathbf{R}^n$  the gradient of the function  $\langle p, d \rangle - x \cdot \ln(p \cdot d)$ , where now we let  $p$  and  $d$  run over the entire space  $\mathbf{R}^n$ , is the vector  $(d - x \cdot p^{-1}, p - x \cdot d^{-1})$  as one readily checks by partial differentiation with respect to the variables  $p_1, \dots, p_n, d_1, \dots, d_n$  and by using the shortened notation introduced above. Therefore, taking into account the constraints  $p \in P$  and  $d \in D$  in  $(Q_x)$  it follows that the theorem of Fermat gives the following conditions for optimality:

$$\begin{aligned} \langle d - x \cdot p^{-1}, \tilde{p} \rangle &= 0 \quad \forall \tilde{p} \in \tilde{P}, \\ \langle p - x \cdot d^{-1}, \tilde{d} \rangle &= 0 \quad \forall \tilde{d} \in \tilde{D}. \end{aligned}$$

These conditions can be rewritten as

$$\begin{aligned} \langle p^{\frac{1}{2}} \cdot d^{\frac{1}{2}} - x \cdot p^{-\frac{1}{2}} \cdot d^{-\frac{1}{2}}, p^{-\frac{1}{2}} \cdot d^{\frac{1}{2}} \cdot \tilde{p} \rangle &= 0 \quad \forall \tilde{p} \in \tilde{P}, \\ \langle p^{\frac{1}{2}} \cdot d^{\frac{1}{2}} - x \cdot p^{-\frac{1}{2}} \cdot d^{-\frac{1}{2}}, p^{\frac{1}{2}} \cdot d^{-\frac{1}{2}} \cdot \tilde{d} \rangle &= 0 \quad \forall \tilde{d} \in \tilde{D}. \end{aligned}$$

Now  $p^{-\frac{1}{2}} \cdot d^{\frac{1}{2}} \cdot \tilde{P}$  and  $p^{\frac{1}{2}} \cdot d^{-\frac{1}{2}} \cdot \tilde{D}$  are orthogonal complements as  $\tilde{P}$  and  $\tilde{D}$  are orthogonal complements. Therefore the conditions above are equivalent to  $p^{\frac{1}{2}} \cdot d^{\frac{1}{2}} - x \cdot p^{-\frac{1}{2}} \cdot d^{-\frac{1}{2}} = 0$ . That is,  $p \cdot d = x$ .

The following reformulation of our results about the problems  $(Q_x)$  is suggestive for our present purpose of finding an  $\varepsilon$ -solution of  $(Q)$ . The Hadamard product establishes a bijection between the set of strictly feasible pairs  $(p, d)$  of  $(Q)$  — that is  $p \in P, p > 0, d \in D, d > 0$  — and the set of positive vectors  $x$  in  $\mathbf{R}^n$ : we let  $(p, d)$  and  $x$  correspond precisely if  $x = p \cdot d$ . Then we write  $p = p(x)$  and  $d = d(x)$ . In figure 6 this bijection is illustrated for  $n = 2$ . If we view the lines  $P$  and  $D$  in  $\mathbf{R}^2$  as coordinate-axes, then the pairs of positive vectors  $(p, d)$  with  $p \in P$  and  $d \in D$  form a region in the ' $P$ - $D$ -plane'. The Hadamard product maps this region bijectively to the strictly positive first quadrant of  $\mathbf{R}^2$ .

It is easy to calculate this bijection in one direction: to  $(p, d)$  one associates the Hadamard product  $p \cdot d$ . If only the inverse would be as easy to calculate, we could find an  $\varepsilon$ -solution of  $(Q)$  by choosing a positive vector  $x$  with  $\|x\|_1 < \varepsilon$  and calculating  $(p(x), d(x))$ ; this is an  $\varepsilon$ -solution of  $Q$  as  $\langle p(x), d(x) \rangle = \|x\|_1$ . This follows by summing the relations  $p_i(x) d_i(x) = x_i$  for  $i = 1, \dots, n$ . As it is, all we can do in this direction is the following: if we have for some positive vector  $x \in \mathbf{R}^n$  a good approximation  $(p_x, d_x)$  of  $(p(x), d(x))$  then we can easily determine a good approximation  $(p_y, d_y)$  of  $(p(y), d(y))$  for any given positive vector  $y$  which is not 'too far away' from  $x$ . This can be done as follows. Apply

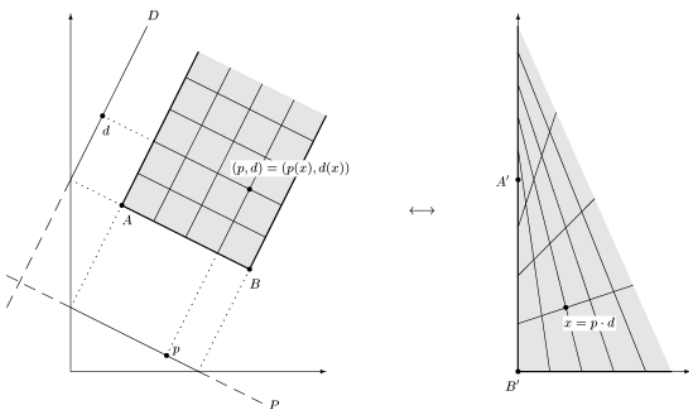


Figure 6 The primal-dual strictly feasible region transformed into the first strict quadrant

one step of the Newton-Raphson algorithm with starting point  $(p_x, d_x)$  to the system

$$\begin{cases} p \in P, \\ d \in D, \\ p \cdot d = y. \end{cases}$$

As  $x$  and  $y$  are not too far away,  $(p(x), d(x))$  and  $(p(y), d(y))$  are not too far away and so  $(p_x, d_x)$  is not too far away from the unique solution  $(p(y), d(y))$  of the system. Therefore the result of this step will be a good approximation of  $(p(y), d(y))$ .

Now suppose that we are so lucky to be in possession of a strictly feasible pair  $(\bar{p}, \bar{d})$  of  $(Q)$ . Then we calculate  $\bar{x} = \bar{p} \cdot \bar{d}$  and view  $(\bar{p}, \bar{d})$  as a good approximation of  $(p(\bar{x}), d(\bar{x}))$ : in fact  $(\bar{p}, \bar{d}) = (p(\bar{x}), d(\bar{x}))$ . Then one can also calculate a good approximation of  $(p(x), d(x))$  for any positive vector  $x$ : by repeated use of the procedure above, moving gradually from  $\bar{x}$  to  $x$ . If  $x$  is chosen such that  $\|x\|_1 < \frac{1}{2}\epsilon$  then the result of this is an  $\epsilon$ -solution for  $(Q)$ . Now the efficiency question arises: how to find strategies to move from  $\bar{x}$  to such a vector  $x$  in as few steps as possible? It would carry us too far to include a full discussion of this question. Figure 7 contains the idea of a strategy which is very efficient both in theory and in practice. Assume that  $\bar{x} = \bar{p} \cdot \bar{d}$  lies on the 45°-line. Then we can try to move gradually from  $\bar{x}$  to a small positive vector  $x$  on the 45°-line by following closely this 45°-line. This line can be parametrized by  $t\bar{x}$  where  $t$  runs from 1 to almost 0. Then the approximations  $(p, d)$  follow closely the path  $(p(t\bar{x}), d(t\bar{x}))$  where  $t$  runs from 1 to almost 0. This path is usually called the *central path*.

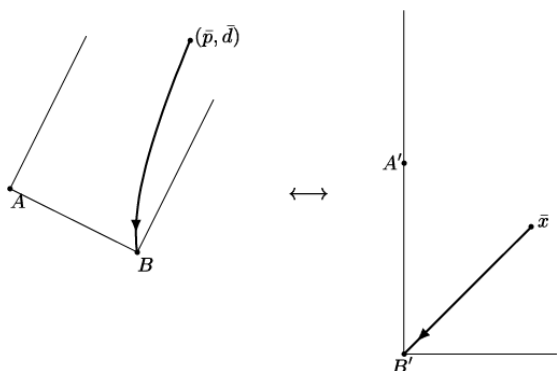


Figure 7 The royal road to the solution: the central path

The computergame Schiet Op [4] allows one to do some simple experiments with this algorithmic idea for ‘toy-problems’ (the case  $n = 2$ ). For a state of the art implementation of interior point methods (also for many nonlinear programming problems) we refer to Sedumi [5].

### Lagrange’s theorem

Lagrange discovered a general method to deal with problems having equality constraints. Let us recall the formulation of Lagrange of his multiplier rule (in [6]).

“One can state the following general principle. If one is looking for the maximum or minimum of some function of many variables subject to the condition that these variables are related by a constraint given by one or more equations, then one should add to the function whose extremum is sought the functions that yield the constraint equations each multiplied by undetermined multipliers and seek the maximum or minimum of the resulting sum as if the variables were independent. The resulting equations, combined with the constraint equations, will serve to determine all unknowns.”

The only essential addition one would like to make to this sentence nowadays is that one should also introduce a multiplier for the objective function. Lagrange’s theorem can be derived from Fermat’s theorem by using the implicit function theorem. Therefore it does not offer anything essentially new. However for practical purposes it is very handy as the following application shows.

### Each symmetric matrix has an orthonormal basis of eigenvectors

Let  $A$  be a symmetric  $n \times n$ -matrix. The problem to maximize  $x^T A x$  subject to  $x^T x = 1$  has a solution  $f_1$  by the theorem of Weierstrass. From the Lagrange multiplier rule it follows immediately that  $A f_1 = \lambda_1 f_1$  for some number  $\lambda_1$ . The problem to maximize  $x^T A x$  subject to  $x^T x = 1$  and  $f_1^T x = 0$  has a solution  $f_2$  by the theorem of Weierstrass. From the Lagrange multiplier rule one readily finds that  $A f_2 = \lambda_2 f_2$  for some number  $\lambda_2$  and so on. As a result we obtain an orthonormal basis  $\{f_i\}_{i=1}^n$  of eigenvectors of  $A$ .

### Inequalities

Lagrange’s multiplier rule can be used to prove all inequalities in a finite number of real variables from [7] by one and the same straightforward method. Let us illustrate this with a simple example.

**Cauchy-Schwarz.** One has

$$x_1 y_1 + \dots + x_n y_n \leq (x_1^2 + \dots + x_n^2)^{\frac{1}{2}} (y_1^2 + \dots + y_n^2)^{\frac{1}{2}},$$

and equality holds precisely if the vectors  $(x_1 \dots x_n)$  and  $(y_1 \dots y_n)$  are linearly dependent.

**Proof.** By homogeneity it suffices to prove that the problem to maximize  $x^T y$  subject to  $x, y \in \mathbf{R}^n$ ,  $x^T x = y^T y = 1$  has as solutions precisely all feasible vectors  $(x, y)$  with  $y = x$ . By Weierstrass’ theorem this problem has a solution. Lagrange’s multiplier rule gives that for each solution  $(\hat{x}, \hat{y})$  there exist numbers  $\lambda_0, \lambda_1, \lambda_2$ , not all zero, such that  $(\hat{x}, \hat{y})$  is a stationary point of the

Lagrange function  $\mathcal{L}(x, y) = \lambda_0 x^T y + \lambda_1 (x^T x - 1) + \lambda_2 (y^T y - 1)$ . That is,

$$0 = \mathcal{L}_x(\hat{x}, \hat{y}) = \lambda_0 \hat{y} + 2\lambda_1 \hat{x} \text{ and}$$

$$0 = \mathcal{L}_y(\hat{x}, \hat{y}) = \lambda_0 \hat{x} + 2\lambda_2 \hat{y}.$$

So  $\hat{x}$  and  $\hat{y}$  are linearly dependent. Using the feasibility of  $(\hat{x}, \hat{y})$  we get  $\hat{y} = \pm \hat{x}$ . The maximality of  $(\hat{x}, \hat{y})$  gives that  $\hat{y} = \hat{x}$ .  $\square$

We recall that the usual proof of Cauchy-Schwarz is based on a little trick.

**The theorem of Karush-Kuhn-Tucker**

For problems with inequality constraints such as the problem to minimize  $f(x_1, x_2)$  subject to  $g(x_1, x_2) \leq 0$ , where  $f$  and  $g$  are differentiable, all solutions  $(\hat{x}_1, \hat{x}_2)$  satisfy the so called Karush-Kuhn-Tucker (KKT) conditions. For this problem one gets that there exist numbers  $\lambda_0, \lambda_1$ , not both zero, such that

1.  $\hat{x}$  is stationary for the Lagrange function  $\mathcal{L}(x) = \lambda_0 f(x) + \lambda_1 g(x)$ ,
2.  $\lambda_0, \lambda_1 \geq 0$ ,
3.  $\lambda_1 g(\hat{x}) = 0$ .

Moreover if  $f$  and  $g$  are convex functions and  $\lambda_0 > 0$ , then these conditions are not only necessary but also sufficient for optimality of  $\hat{x}$ . The KKT-conditions for this problem can be derived from the Lagrange multiplier rule. For this one should distinguish two cases:

1. The constraint is binding:  $g(\hat{x}) = 0$ . Then the KKT-conditions follow immediately from the Lagrange multiplier rule for the problem  $f(x) \rightarrow \min$  subject to  $g(x) = 0$ .
2. The constraint is not binding:  $g(\hat{x}) < 0$ . Then the KKT-conditions follow immediately from Fermat's theorem for the problem  $f(x) \rightarrow \min$  subject to  $g(x) < 0$ .

Therefore the KKT-conditions do not offer anything essentially new. However it is convenient to use them.

Having seen this special case, one can easily guess the correct form of the KKT-conditions for minimizing functions of several variables with a finite number of equality and inequality constraints and derive them from the Lagrange multiplier rule and Fermat's theorem.

**Zero-sum games**

Many games between two persons can be modeled as follows. Let  $M$  be an  $m \times n$ -matrix. Person 1 can choose between  $m$  moves and simultaneously person 2 can choose between  $n$  moves. If person 1 chooses  $i$  and person 2 chooses  $j$  then person 1 has to pay  $m_{ij}$  euro to person 2. If  $m_{ij}$  is negative, then this has the natural interpretation: person 2 has to pay  $-m_{ij} = |m_{ij}|$  euro to person 1. The game is to be played repeatedly. The question is what is the best strategy for each player? 'Best' means here highest guaranteed expected payoff. We allow the following type of strategy. A strategy for player 1 can be described by a vector  $p$  in the set  $P = \{p \in \mathbf{R}^m \mid p \geq 0 \text{ and } \sum_{i=1}^m p_i = 1\}$ . The meaning of this is that person 1 chooses move  $i$  with chance  $p_i$  for all  $i$ . Similarly

a strategy for player 2 can be described by a vector  $q$  in the set  $Q = \{q \in \mathbf{R}^n \mid q \geq 0 \text{ and } \sum_{j=1}^n q_j = 1\}$ . Then the expected payoff is  $p^T M q$ .

By using the KKT-conditions for LP-problems one can derive the following theorem of von Neumann. There exists a Nash-equilibrium, that is,  $\hat{p} \in P$  and  $\hat{q} \in Q$  such that  $p^T M \hat{q} \leq \hat{p}^T M \hat{q} \leq \hat{p}^T M q$  for all  $p \in P$  and  $q \in Q$ . That is, if person 1 chooses  $\hat{p}$  and person 2 chooses  $\hat{q}$ , then neither of them is tempted to choose another strategy.

**Euler's equation and a transversality condition**

There are many interesting optimization problems where the variable  $x$  which has to be chosen optimally is not a quantity  $x \in \mathbf{R}$  or a finite number of quantities  $x \in \mathbf{R}^n$  but a continuously differentiable function  $x(t)$  of one variable  $t$  on an interval  $[t_0, t_1]$ , that is  $x(\cdot) \in C^1[t_0, t_1]$ . Many of these problems can be modeled as follows: minimize

$$J(x(\cdot)) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t)) dt$$

where  $x(\cdot)$  runs over  $C^1[t_0, t_1]$ . Here  $t_0, t_1 \in \mathbf{R}$  with  $t_0 < t_1$ , the function  $f$  on  $\mathbf{R}^3$  is continuous and  $\dot{x}(t)$  is the derivative of the function  $x(t)$ . Euler discovered that  $\hat{f}_x - \frac{d}{dt} \hat{f}_{\dot{x}} = 0$  (Euler's equation) and  $\hat{f}_x(t_0) = \hat{f}_x(t_1) = 0$  (transversality conditions) for all solutions  $\hat{x}(\cdot)$  of this problem. In a more precise notation the Euler equation is

$$\frac{\partial f}{\partial x}(t, \hat{x}(t), \dot{\hat{x}}(t)) - \frac{d}{dt} \left[ \frac{\partial f}{\partial \dot{x}}(t, \hat{x}(t), \dot{\hat{x}}(t)) \right] = 0 \quad \forall t \in [t_0, t_1]$$

and the transversality conditions are

$$\frac{\partial f}{\partial \dot{x}}(t_0, \hat{x}(t_0), \dot{\hat{x}}(t_0)) = \frac{\partial f}{\partial \dot{x}}(t_1, \hat{x}(t_1), \dot{\hat{x}}(t_1)) = 0.$$

At first sight this looks like a completely new method. However we shall now make plausible that it is just Fermat's theorem; it is routine to turn this plausibility argument into an exact proof.

The derivative  $J'(\hat{x}(\cdot))$  of  $J$  in  $\hat{x}(\cdot)$  is defined to be the linear function on  $C^1[t_0, t_1]$  for which, loosely speaking,

$$J(\hat{x} + h) - J(\hat{x}) \approx J'(\hat{x})(h)$$

for all  $h \in C^1[t_0, t_1]$  for which  $|h(t)|$  and  $|\dot{h}(t)|$  are sufficiently small for all  $t \in [t_0, t_1]$ . To be more precise,

$$J(\hat{x} + h) = J(\hat{x}) + J'(\hat{x})(h) + o(h), \quad h \rightarrow 0$$

in the normed vectorspace  $C^1[t_0, t_1]$  with norm defined by  $\|f\|_{C^1} = \max(\sup_{t \in [t_0, t_1]} |f(t)|, \sup_{t \in [t_0, t_1]} |\dot{f}(t)|)$ .

Now we will 'derive' the following explicit formula for  $J'(\hat{x})$ :

$$J'(\hat{x})(h) = \int_{t_0}^{t_1} (\hat{f}_x - \frac{d}{dt} \hat{f}_{\dot{x}}) h dt + [\hat{f}_{\dot{x}} h]_{t_0}^{t_1} \quad \text{for all } h \in C^1[t_0, t_1]. \quad (*)$$

The difference  $J(\hat{x} + h) - J(\hat{x})$  equals by definition

$$\int_{t_0}^{t_1} [f(t, \hat{x} + h, \dot{\hat{x}} + \dot{h}) - f(t, \hat{x}, \dot{\hat{x}})] dt.$$

If  $\|h\|_{C^1}$  is sufficiently small this is ‘after linearization of the integrand’  $\approx \int_{t_0}^{t_1} [\hat{f}_x h + \hat{f}_x' h] dt$ . By partial integration this can be rewritten as

$$\int_{t_0}^{t_1} (\hat{f}_x - \frac{d}{dt} \hat{f}_x') h dt + [\hat{f}_x' h]_{t_0}^{t_1}.$$

Now we observe that this expression is linear in  $h$ . This finishes the ‘derivation’ of (\*). Thus prepared we show that the result of Euler is essentially Fermat’s theorem, that is, it is equivalent to  $J'(\hat{x}) = 0$ .

Well, the explicit formula (\*) for  $J'(\hat{x})$  makes it possible to decode the condition  $J'(\hat{x}) = 0$ . As the function  $h \in C^1[t_0, t_1]$  in (\*) is arbitrary, it ‘follows’ that  $\hat{f}_x - \frac{d}{dt} \hat{f}_x' = 0$  and  $\hat{f}_x'(t_0) = \hat{f}_x'(t_1) = 0$ .

Finally we give a variant of the result of this section. If we add the equality constraints  $x(t_0) = x_0$  and  $x(t_1) = x_1$  to the problem, then each solution  $\hat{x}(\cdot)$  satisfies only the Euler equation and not necessarily the transversality conditions. This result can be derived from the result above in the same way as the Lagrange multiplier rule can be derived from Fermat’s theorem.

**Growth theory and Ramsey’s model**

How much should a nation save? Two possible answers are: *nothing* (“Après nous le déluge”, Louis XV) and *everything* (“Yes, they live on rations, they deny themselves everything... But with this gold new factories will be built ... a guarantee for future plentifulness” from the novel ‘Children of the Arbat’ of A. Rybakov [8] (p. 34), illustrating the policy of Stalin). A third answer is given by Ramsey’s model: choose the golden middle road; save something, but consume (enjoy) something as well. Ramsey’s paper [9] on the optimal social saving behaviour is among the very first applications of the calculus of variations to economics. This paper has exerted an enormous if delayed influence on the current literature on optimal economic growth. A simple version of this model is the following optimization problem.

$$I(C(\cdot), k(\cdot)) = \int_0^\infty U(C)e^{-\theta t} dt \rightarrow \max$$

subject to  $\dot{k} = F(k) - C$ .

Here

- $C = C(t)$  = the rate of consumption at time  $t$ ,
- $U(C)$  = the utility of consumption  $C$ ,
- $\theta$  = the discount rate,
- $k = k(t)$  = the capital stock at time  $t$ ,
- $F(k)$  = the rate of production when the capital stock is  $k$ .

It is usual to assume  $U(C) = \frac{C^{1-\rho}}{1-\rho}$  for some  $\rho \in (0, 1)$  and  $F(k) = Ak^{\frac{1}{2}}$  for some positive constant  $A$ . Then the solution of the problem cannot be given explicitly; however a qualitative analysis shows that it is optimal to let consumption grow asymptotically to some finite level. Now let us consider a modern variant of this model from [10] and [11]. The intuition behind the model above allows one to model the production function as  $F(k) = Ak$  for some positive constant  $A$  instead of  $F(k) = Ak^{\frac{1}{2}}$ . Now we apply Euler’s result to this problem. To this end we eliminate  $C$ ; the result is the problem

$$J(k(\cdot)) = \int_0^\infty -\frac{(Ak - \dot{k})^{1-\rho}}{1-\rho} e^{-\theta t} dt \rightarrow \min.$$

Let  $\hat{k}(\cdot)$  be a solution of this problem and write  $\hat{C}(\cdot)$  for the corresponding consumption function. The Euler equation gives

$$-A\hat{C}^{-\rho} e^{-\theta t} - \frac{d}{dt} (\hat{C}^{-\rho} e^{-\theta t}) = 0.$$

This implies  $\hat{C}^{-\rho} e^{-\theta t} = r e^{-At}$  for some constant  $r$ . Therefore

$$\hat{C} = C_0 e^{\frac{A-\theta}{\rho} t}.$$

Therefore this modern version has a more upbeat conclusion: there is an explicit formula for the solution of the problem and moreover consumption can continue to grow forever to unlimited levels.

**Pontrijagin’s Maximum Principle**

The result of Euler, mentioned above, turned out to be very flexible and has led to the creation of the Calculus of Variations. Many types of optimization problems where the variable which has to be chosen optimally is a function  $x(t)$  of one variable  $t$  have been analyzed with success with variants of this method. However, around the middle of the 20th century engineers encountered problems which could not be treated with any variant of this method. The reason is that constraints of the type ‘ $\dot{x}(t) \in U$  for all  $t$ ’ where  $U$  is some given subset of  $\mathbf{R}$ , could not be made to fit into the framework of the Calculus of Variations. Then in 1953 Pontrijagin and his coworkers succeeded in overcoming this problem, by proposing what seemed to be an entirely new method. Consider for example the following problem

$$J(x(\cdot)) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t)) dt \rightarrow \min$$

subject to  $x(\cdot) \in KC[t_0, t_1]$  and  $\dot{x}(t) \in U \forall t \in [t_0, t_1]$

where  $x(\cdot)$  is differentiable.

Here  $t_0, t_1$  are given real numbers with  $t_0 < t_1$ ,  $f$  is a continuous function on  $\mathbf{R}^3$  and  $U$  is a given subset of  $\mathbf{R}$ . We recall that  $KC[t_0, t_1]$  consists of all continuous, piecewise continuous differentiable functions on  $[t_0, t_1]$ , with at most finitely many kinks; these kinks must be nice in the sense that left- and rightderivative must exist. The Hamilton function is defined by

$$H = H(t, x, u, \lambda_0, p) = pu - \lambda_0 f(t, x, u).$$

The result is that for each solution  $\hat{x}(\cdot)$  of this problem there exists  $\hat{\lambda}_0 \in [0, \infty)$  and  $\hat{p}(\cdot) \in KC^1[t_0, t_1]$  not both zero such that the following conditions hold:

$$\begin{aligned} \dot{\hat{x}} &= \hat{H}_p, \\ \dot{\hat{p}} &= -\hat{H}_x, \\ \hat{H}(t) &= \max_{u \in U} H(t, \hat{x}(t), u, \hat{p}(t), \hat{\lambda}_0), \\ \hat{p}(t_0) &= \hat{p}(t_1) = 0. \end{aligned}$$

Here we use the same conventions to shorten the notation as before. Just as the result of Euler, this one — called Pontrijagin’s Maximum Principle (PMP) — turned out to be very flexible. It has led to the creation of the Optimal Control Theory [12]. Below we give one of the many applications to mathematical analysis, science and economics, of one of the variants of PMP.

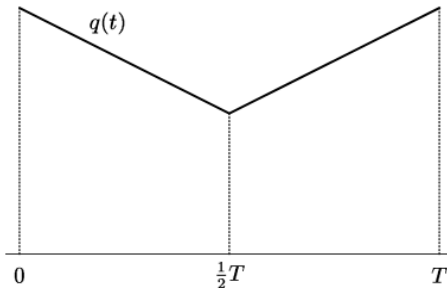


Figure 8 Forecast for the development of the price as predicted by the trader

**Commodity trading**

Let us consider the buying and selling of a commodity by traders who do not intend to use the commodity themselves. The skill of a successful trader depends on the ability to make an accurate forecast for the development of the price in the future. Given a forecast, it is possible to pose an optimal control problem to determine when the commodity should be bought or sold and when the trader should be inactive. In practice the operations of buying and selling will be discrete, but here we use a continuous model from [13]; this is easier to use and gives the same insight as a discrete model.

$$\begin{aligned}
 J(x_1(\cdot), x_2(\cdot), u(\cdot)) &= -x_1(T) - q(T)x_2(T) \rightarrow \min \\
 \text{subject to } x_1(\cdot), x_2(\cdot) &\in KC^1[0, T], u(\cdot) \in KC^0[0, T], \\
 \dot{x}_1 &= qu - sx_2, \dot{x}_2 = -u, x_1(0) = X, x_2(0) = 0, \\
 u(t) &\in [-1, 1] \text{ for all points of continuity of } u(\cdot).
 \end{aligned}$$

Here,

- $T$  = the time period for which the trader predicts the price
- $x_1(t)$  = the amount of cash which is held at time  $t$
- $x_2(t)$  = the amount of the commodity held at time  $t$
- $q(t)$  = the price of the commodity at time  $t$  as predicted by the trader; in the problem (P) the function  $q(\cdot)$  ('the forecast') is considered as given
- $u(t)$  = the selling rate at time  $t$ ; negative values of  $u$  correspond to a buying phase
- $X$  = the amount of cash held at time 0 ('now').

The goal is to maximize the total value of the assets at time  $T$ . If we apply the appropriate variant of PMP to the problem, then we get, for each given function  $q(t)$ , the optimal trading strategy. Let us consider the forecast from figure 8.

Then the optimal strategy turns out to be 'governed' by the following so-called shadowprice-function  $\hat{p}(t) = s(t - T) + q(T)$ . Figure 9 contains the graphs of both the forecasted price  $q(\cdot)$  and the shadowprice  $\hat{p}(\cdot)$ .

From  $t = 0$  till  $t = t_s$  ("the switching time") the price  $q(t)$  is higher than the shadowprice  $\hat{p}(t)$  and the trader should sell as fast as possible, from  $t = t_s$  till  $t = T$  the price  $q(t)$  is lower than  $\hat{p}(t)$  and the trader should buy as fast as possible. For other forecasts  $q(\cdot)$  one can also have periods that it is optimal to be inactive. Furthermore we point out that the shadowprice function  $\hat{p}(t)$  which plays such a crucial role in the optimal strategy 'is' the function  $\hat{p}(\cdot)$  occurring in PMP, provided that one chooses  $\lambda_0 = 1$ .

Finally we take a critical look at this model. We have not restricted the amount of the commodity held  $x_2$  to be nonnegative:

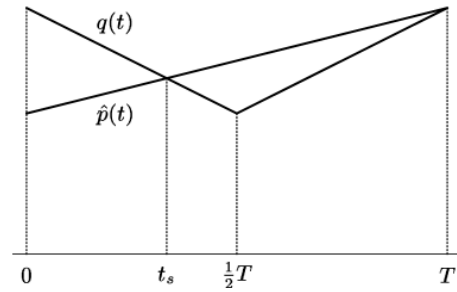


Figure 9 The shadowprice  $\hat{p}(t)$  warns the trader to take action before the price reaches its bottom

we allow short-selling. That is, selling of goods that are not actually in the trader's possession. This actually occurs in the example above. It is only when  $t > 2t_s$  that the trader actually possesses any of the commodity. Here two views are possible. Either one forbids short-selling, by introducing the constraint  $x_2(t) \geq 0$  for all  $t$ . Then it turns out that the optimal profit is halved. Or one allows short-selling; then the model above has a flaw which should be corrected: there is short-selling, the negative value of  $x_2$  implies that the storage charge produces a profit, which is not very realistic.

**Unification in the style of Lagrange**

In retrospect one can see PMP as a realization of the ideal of Lagrange, as Tikhomirov has shown (for example in [14]). To clarify this, we now discuss a problem from Newton's Principia [15]: "figures may be compared together as to their resistance; and those may be found which are most apt to continue their motions in resisting mediums". Newton proposed a solution which was however not understood until recently; it has generally been considered as an example of a mistake by a genius. One of the formalizations is the following.

$$\begin{aligned}
 (P) \quad J(u(\cdot)) &= \int_0^T \frac{tdt}{1+u^2} \rightarrow \min \\
 \text{subject to } u(\cdot) &\in KC[0, T], \int_0^T u dt = \xi, u \geq 0.
 \end{aligned}$$

The relation of problem (P) with Newton's problem can be described as follows. Let  $\hat{u}(\cdot)$  be a solution of (P); take the primitive  $\hat{x}(\cdot)$  of  $\hat{u}(\cdot)$  which has  $\hat{x}(0) = 0$ . Its graph is a curve in the  $t$ - $x$ -plane. Now we take the surface of revolution of this curve around the  $x$ -axis. This is precisely the shape of the front of the optimal figure in Newton's problem. The details of this relation are given in [14]. The constraint  $u \geq 0$  (the monotonicity of  $x(\cdot)$ ) was not made explicit by Newton. We stress once more that precisely this type of constraint can be dealt with very well by PMP, but not by the Calculus of Variations. It is natural to interpret Lagrange's method for this problem as follows. There are constants  $\lambda_0 \geq 0$  and  $\lambda$ , not both zero, such that each solution  $\hat{u}(\cdot)$  of (P) is a solution of the following auxiliary problem

$$\begin{aligned}
 (Q) \quad I(u(\cdot)) &= \int_0^T \frac{tdt}{1+u^2} + \lambda \left[ \int_0^T u dt - \xi \right] \rightarrow \min \\
 \text{subject to } u &\in KC[0, T], u \geq 0.
 \end{aligned}$$

It is intuitively clear that a piecewise continuous function  $\hat{u}(\cdot)$



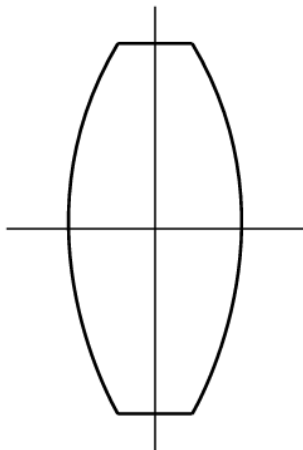


Figure 10 The optimal shape of spacecraft as proposed by Newton

is a solution of (Q) precisely if for all points  $t$  of continuity of  $\hat{u}(\cdot)$  the nonnegative value of  $u$  which minimizes the integrand  $g_t(u) = \lambda_0 \frac{t}{1+u^2} + \lambda u$  is  $u = \hat{u}(t)$ . In fact it is not difficult to give a rigorous proof of this claim. Thus the problem has been reduced to the minimization of differentiable functions  $g_t(u)$  of a nonnegative variable  $u$ . Clearly for each  $t$  the function  $g_t(u)$  is minimal either at  $u = 0$  or at a solution of the stationarity equation  $\frac{d}{du} g_t(u) = 0$ . Now a straightforward calculation leads to an explicit determination of the — unique — solution of the problem (Q). One can verify directly that this is also a solution of (P). The resulting optimal shape is given in figure 10 (in cross-section). We observe in particular that it has kinks.

This is precisely the solution which was proposed by Newton. The method of solution above is essentially the same as the one by PMP, as one can verify without difficulty. Also in another respect Newton was ahead of his time here: his solution has been used to design the optimal shape of spacecraft.

Not only PMP can be seen as a realization of the idea of Lagrange. Tikhomirov and Ioffe have realized — for example in [16] — the idea of Lagrange for an extensive class of so-called mixed problems. Here ‘mixed’ means that the structure of all ingredients of the problem is a mixture of convexity and smoothness. The result is a unification of almost all the known and unknown necessary conditions which are used to solve optimization problems.

Let us explain the addition of ‘and unknown’. For certain problems of interest the unification allows to write down conditions, although the necessity of these conditions is not known to hold. Then an analysis of these conditions leads to certain concrete ‘candidate solutions’ for our problem. So far this is a completely heuristic method, the result of which can be viewed perhaps as ‘a solution of the problem for commercial purposes’. However once one has a concrete candidate it is usually possible to obtain somehow mathematical certainty that one has indeed a solution of the problem. We refer to the paper [17] for a number of convincing examples of this strategy.

**Unification in the style of Fermat**

Finally we sketch the idea of a new, geometric unification of the necessary conditions, which is in the style of Fermat’s method: put the derivative equal to zero. We shall begin with two special

cases, smooth problems and convex problems, before we consider general mixed smooth-convex problems.

**Smooth problems**

Consider to begin with the simplest type of unconstrained problem

$$(P) \quad f(x) \rightarrow \min \quad \text{subject to } x \in \mathbf{R},$$

where  $f$  is a differentiable function on  $\mathbf{R}$ . The tangent to the graph of  $f$  at a point  $\hat{x}$  is the graph of an affine function  $L_f$ . Explicitly  $L_f(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$ , the linear approximation of  $f$  at  $\hat{x}$ . The graphs of  $f$  and  $L_f$  are given in figure 11.

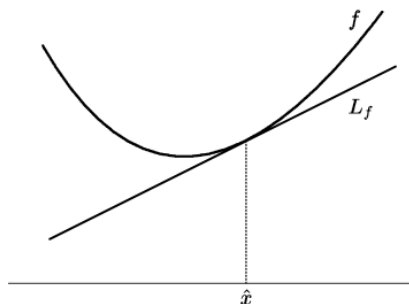


Figure 11 Smooth linearization of a function

The theorem of Fermat states that  $f'(\hat{x}) = 0$  if  $\hat{x}$  is a solution of (P). Observe that  $f'(\hat{x}) = 0$  precisely if the function  $L_f$  is constant; moreover a constant function is minimal everywhere. This suggests the following reformulation of the theorem of Fermat:

$$\hat{x} \text{ is a solution of (P)} \Rightarrow \hat{x} \text{ is a solution of (L).}$$

Here (L) is defined to be the following linearization of the problem (P) at  $\hat{x}$

$$(L) \quad L_f(x) \rightarrow \min \quad \text{subject to } x \in \mathbf{R}.$$

This way to view the theorem of Fermat is illustrated in figure 12.

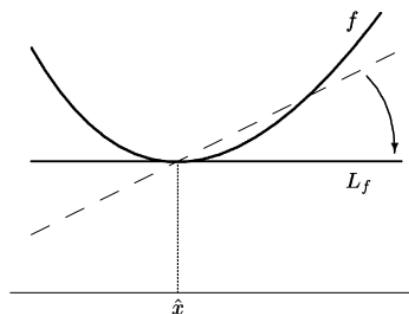


Figure 12 Fermat’s method for smooth problems

As a second example we consider the simplest type of problem with an equality constraint

$$(P') \quad f(x) \rightarrow \min \quad \text{subject to } g(x) = 0,$$

where  $f$  and  $g$  are differentiable functions on  $\mathbf{R}^2$ .

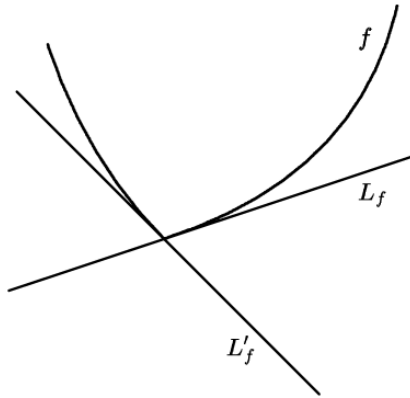


Figure 13 Non-uniqueness of convex linearizations

Let  $L_f$  (respectively  $L_g$ ) be the linear approximation of  $f$  (respectively  $g$ ). Consider the following ‘linearization’ of the problem  $(P')$  at  $\hat{x}$ .

$$(L') \quad L_f(x) \rightarrow \min \quad \text{subject to } L_g(x) = 0.$$

The Lagrange multiplier rule can be reformulated as follows (provided that the gradient  $g'(\hat{x})$  is nonzero)

$$\hat{x} \text{ is a solution of } (P') \Rightarrow \hat{x} \text{ is a solution of } (L').$$

Let us verify this. One has  $g(\hat{x}) = 0$  as  $\hat{x}$  is feasible for  $(P')$ . The Lagrange multiplier rule states that there exists  $\lambda \in \mathbf{R}$  with  $f'(\hat{x}) + \lambda g'(\hat{x}) = 0$  provided that  $g'(\hat{x}) \neq 0$ . For this condition one has the following equivalent ‘dual’ description: one has  $f'(\hat{x})(x - \hat{x}) = 0$  for all  $x \in \mathbf{R}^2$  with  $g'(\hat{x})(x - \hat{x}) = 0$ . That is, the function  $L_f(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$  is constant on the zero-set of the function  $L_g(x) = g'(\hat{x})(x - \hat{x})$ . This finishes the verification of the equivalence of the two formulations.

More generally one can produce — necessary — conditions for all smooth optimization problems in essentially the same way. Here problems are called smooth if they are of the following type

$$f(x) \rightarrow \min \quad \text{subject to } g_1(x) = \dots = g_m(x) = 0,$$

where  $f, g_1, \dots, g_m$  are differentiable functions on an open subset of a normed vectorspace.

Two happy circumstances are responsible for the success of conditions ‘in the style of Fermat’ for smooth optimization problems.

1. Possibility. The concept tangent space allows one to define smooth linearization for functions defined on a subset of a normed vectorspace.
2. Effectiveness. One can develop a calculus to compute tangent spaces in favorable situations. Indeed the theorem of the tangent-space of Lyusternik (a version of the implicit function theorem) reduces the computation of tangent-spaces to differential calculus.

**Convex problems**

An optimization problem is called convex if it is of the following form

$$(P'') \quad f(x) \rightarrow \min \quad \text{subject to } x \in C,$$

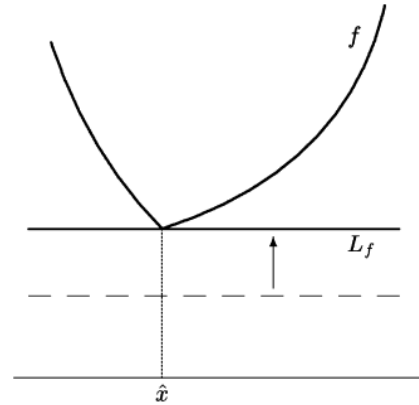


Figure 14 Fermat's method for convex problems

where  $C$  is a convex subset of a vectorspace  $X$  and  $f$  is a convex function on  $C$ . Now we assume that the following regularity condition holds for some  $\hat{x} \in C$ : for each  $x \in C$  the limit of  $t^{-1}[f(\hat{x} + t(x - \hat{x})) - f(\hat{x})]$  for  $t \downarrow 0$  exists and lies in  $\mathbf{R}$ . Then there exists an affine function  $L_f$  on  $X$  with  $L_f(x) \leq f(x)$  for all  $x \in X$  and  $L_f(\hat{x}) = f(\hat{x})$  ( $L_f$  supports  $f$  at  $\hat{x}$ ). This fact can be derived from the well-known separation theorem for convex sets. The function  $L_f$  is not necessarily unique, as figure 13 shows.

Nevertheless one can consider for each such function  $L_f$  the following problem as a linearization of  $(P'')$  at  $\hat{x}$ .

$$(L'') \quad L_f(x) \rightarrow \min$$

Now we are ready to propose a necessary condition for  $(P'')$  ‘in Fermat’s style’:

$$\hat{x} \text{ is a solution of } (P'') \Rightarrow \hat{x} \text{ is a solution of some } (L'').$$

This implication — which is of course completely trivial — is illustrated in figure 14.

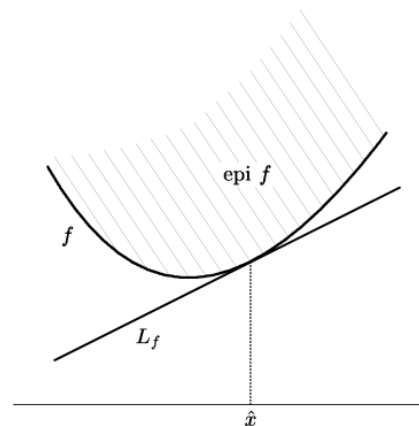


Figure 15 A convex function viewed as a convex set

In order for this necessary condition to be of practical value, one needs a calculus to compute ‘linearizations in the convex sense’. We are now going to show why we can take for this the calculus for computing duals of convex cones — which is well-developed. We recall that a subset  $K$  of a vectorspace is a convex cone if it is closed under multiplication with nonnegative scalars and under addition. Then the dual  $K^*$  of  $K$  is the set of linear functions  $\alpha$  on  $V$  for which  $\alpha(k) \geq 0$  for all  $k \in K$ . This is again a convex cone.

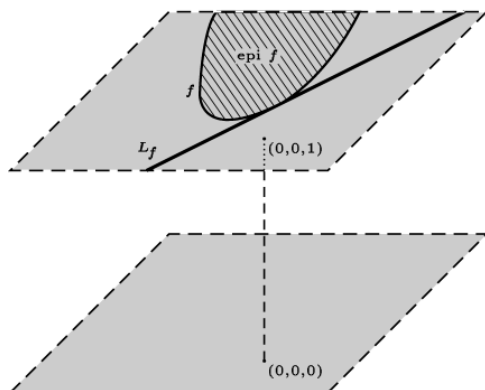


Figure 16 Lifting up figure 15 to level 1

Now let  $C, f, \hat{x}$  and  $L_f$  be as above.

We explain the idea for the special case  $X = \mathbf{R}$ , for the convenience of exposition only. Then the graphs of  $f$  and  $L_f$  lie in the — horizontal — plane  $\mathbf{R}^2$  as is shown in figure 15. We add a vertical dimension and lift this horizontal plane up vertically to level 1. The result is given in figure 16.

Then we form the cone  $K_f$  generated by the lifted up copy of  $\text{epi } f$ , the epigraph of  $f$ , that is,  $K_f = \mathbf{R}^+(\text{epi } f \times 1)$ . This is a convex cone as  $f$  is a convex function. We form the linear subspace  $W_f$  of  $\mathbf{R}^3$  spanned by the lifted up copy of the graph of  $L_f$ , that is,  $W_f = \mathbf{R} \cdot (\text{graph } L_f \times 1)$ .

Then  $W_f$  is a plane in  $\mathbf{R}^3$  through the origin which has the convex cone  $K_f$  on one of its two sides and which contains the point  $(\hat{x}, f(\hat{x}), 1)$  of the convex cone  $K_f$ . Now consider a nonzero linear function  $\beta$  on  $\mathbf{R}^3$  which has kernel equal to  $W_f$ . Then either  $\beta$  or  $-\beta$  lies in the dual of the convex cone  $K_f^*$  by the properties of  $W_f$  above and the definition of the dual of a convex cone. Thus we obtain a nonzero element  $\alpha$  of  $K_f^*$  with  $\alpha(\hat{x}, f(\hat{x}), 1) = 0$ . This element  $\alpha$  is determined up to a positive scalar multiple; that is its ray  $\mathbf{R}^+ \cdot \alpha$  is uniquely determined. Thus we have constructed a map from the set of linearizations  $L_f$  of the convex function  $f$  at  $\hat{x}$  to the set of rays  $\mathbf{R}^+ \cdot \alpha$  of the dual cone  $K_f^*$  with  $\alpha(\hat{x}, f(\hat{x}), 1) = 0$ . It can be shown that this map is a bijection; for this one needs the regularity condition made above. This finishes the sketch of the connection between the problem of linearization of the convex function  $f$  at  $\hat{x}$  and that of the computation of duals of convex cones.

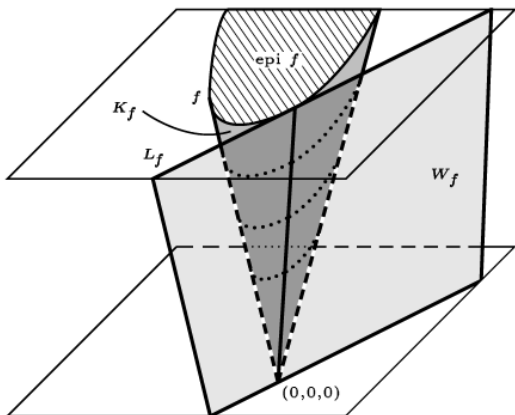


Figure 17 A convex linearization of a function viewed as the element of a dual cone

For convex problems there are, just as for smooth problems, two happy circumstances which are responsible for the success of conditions 'in the style of Fermat'.

1. Possibility. The concept supporting hyperplane allows one to define 'convex linearization' of functions defined on the subset of a vectorspace.
2. Effectiveness. One can develop a calculus to compute supporting hyperplanes in favorable situations. Indeed the separation theorem of convex sets reduces the computation of supporting hyperplanes to the calculus of duals of convex cones in vectorspaces.

**Mixed smooth-convex problems**

Consider the following set-up:  $X, U, Y$  are normed vectorspaces,  $F$  is a function on the product  $X \times U \times Y$  which takes values in the extended real line  $\mathbf{R} = \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$  and vectors  $\hat{x} \in X, \hat{u} \in U$  with  $F(\hat{x}, \hat{u}, 0) \in \mathbf{R}$ . To these ingredients we associate the problem

$$(P''') \quad F(x, u, 0) \rightarrow \min.$$

A mixed smooth-convex linearization  $L$  of  $F$  is defined to be an affine function on  $X \times U \times Y$  such that  $L(x, \hat{u}, y)$  is a smooth linearization of  $F(x, \hat{u}, y)$  at  $(\hat{x}, 0)$  and  $L(\hat{x}, u, y)$  is a convex linearization of  $F(\hat{x}, u, y)$  at  $(\hat{u}, 0)$ . For such a function  $L$  the problem

$$(L''') \quad L(x, u, 0) \rightarrow \min$$

is called a mixed smooth-convex linearization of  $(P''')$ .

I think that under relatively mild assumptions, among these 'smoothness in the variables  $(x, y)$ ' and 'convexity in the variables  $(u, y)$ ', the following implication holds:

$$\begin{aligned} (\hat{x}, \hat{u}) \text{ is a solution of } (P''') \\ \Rightarrow (\hat{x}, \hat{u}) \text{ is a solution for some mixed} \\ \text{smooth-convex linearization } (L'''). \end{aligned}$$

A result of this type is given in [18]. Moreover I think that the analysis of the condition ' $(\hat{x}, \hat{u})$  is a solution for some mixed smooth-convex linearization  $(L''')$ ' is the best practical way of spotting solutions of mixed smooth-convex optimization problems. Here the possibility to make a heuristic use of 'necessary conditions' should be stressed again. Thus for each problem of type  $(P''')$  above, one can write down the condition ' $(\hat{x}, \hat{u})$  is a solution for some mixed smooth-convex linearization  $(L''')$ '. For this one does not have to pay attention to any assumptions. Then one can analyze this condition. Any concrete candidate  $(\hat{x}, \hat{u})$  that turns up in this analysis can usually be checked for optimality without difficulty.

Finally we mention that it is not difficult to derive the unification in Lagrange's style and so in particular Pontrijagin's Maximum Principle from the unification in Fermat's style.

**How to choose  $F$ ?**

Let us consider the simplest example of a problem of mixed smooth-convex type.

$$f(x_1, x_2) \rightarrow \min \quad \text{subject to } g(x_1, x_2) \leq 0,$$

with  $f$  and  $g$  differentiable. Introduce a slack variable  $u \geq 0$  in order to replace the inequality constraint  $g(x) \leq 0$  by the equality constraint  $g(x) + u = 0$ . Then replace the righthandside of this

equality constraint by the parameter  $y$ ; this gives  $g(x) + u = y$ . Then define

$$F(x, u, y) = f(x) \quad \text{if } g(x) + u = y \text{ and } u \geq 0,$$

$$F(x, u, y) = \infty \quad \text{otherwise.}$$

Then the problem to minimize  $F(x, u, 0)$  is equivalent to the original problem. If one works out the 'condition' above for this choice of  $F$ , then one gets the usual KKT-conditions for this problem.

### On optimization: from art to craft

*"The most fascinating pursuit is to follow the thoughts of a great man."*

A.S. Pushkin

*"We must make it our goal to find a method of solution of all problems . . . by means of a single simple method."* D'Alembert

One of the most accessible popular books on optimization is *Stories about Maxima and Minima* by V.M. Tikhomirov [14]. It consists of two parts. The first part is inspired by the words of Pushkin. It offers examples of the *art* of solving special optimization problems by some of the great mathematicians of the past. The second

part is inspired by the words of D'Alembert. It teaches a *craft* which allows anyone who is interested, to solve these and other optimization problems by one straightforward strategy. The main issue is here how to obtain necessary conditions for optima. In the present paper we have given a survey of these conditions and their unification.

Finally we mention some matters which deserve further investigation. The necessity of our condition in the style of Fermat is trivial for the special cases of smooth and convex problems; however it is surprisingly hard to prove for general mixed problems. In fact it has only been proved so far under assumptions which are probably much too strong. In the second place it is of crucial importance for the success of the strategy to have an efficient calculus for computing duals of convex cones. It seems to me that there is room for progress here. Finally the proposed strategy using necessary conditions in the style of Fermat and Lagrange has clear limits. For example for multidimensional variational problems no necessary conditions are known of a strength comparable to that of Pontrijagin's Maximum Principle for one dimensional variational problems. Here something entirely new has to be found. ◀

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