A formula for \( \pi(x) \) applied to a result of Koninck-Ivić

We are going to give an approximate formula for \( \pi(x) \) which is better than the well known \( \pi(x) \sim \frac{x}{\log x} \), or than the more precise formula from [2]: \( \pi(x) \sim \frac{x}{\log x - \gamma} \), meaning that \( \pi(x) = \frac{x}{\log x - \alpha(x)} \), where \( \lim_{x \to \infty} \alpha(x) = 1 \). We will prove

**Theorem 1.**

\[
\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_n (1 + \alpha_n(x))}{\log^n x},
\]

where \( k_1, k_2, \ldots, k_n \) are given by the recurrence relation

\[
k_m + 1! k_{m-1} + 2! k_{m-2} + \ldots + (m-1)! k_1 = n \cdot n!, \quad n = 1, 2, 3, \ldots
\]

and \( \lim_{x \to \infty} \alpha_n(x) = 0 \).

**Proof.** The following asymptotic formula

\[
\pi(x) = \text{Li}(x) + O(x \exp(-a \log x)^n),
\]

where \( a \) and \( \alpha \) are positive constants and \( \alpha < \frac{1}{2} \) is well known [3]. Integrating by parts and taking into account that

\[
x \exp(-a \log x)^\alpha = o\left(\frac{x}{\log^{n+2} x}\right),
\]

where \( n \geq 1 \), it follows that

\[
\pi(x) = x \left(\frac{1}{\log x} + \frac{1!}{\log^2 x} + \ldots + \frac{n!}{\log^{n+1} x}\right) + O\left(\frac{x}{\log^{n+2} x}\right)
\]

(1)

We define the constants \( k_1, k_2, \ldots, k_n \) by the recurrence

\[
k_m + 1! k_{m-1} + 2! k_{m-2} + \ldots + (m-1)! k_1 = (m+1)! - m!,
\]

for \( m = 1, 2, \ldots, n \). For \( y > 0 \) we consider

\[
f(y) = \left(\sum_{i=1}^{n} \frac{i!}{y^{i+1}}\right)(y - 1 - \sum_{i=1}^{n} \frac{k_i}{y^i}),
\]

and we have

\[
f(y) = 1 + \frac{2! - 1! - k_1}{y^2} + \frac{3! - 2! - 1! k_1 - k_2}{y^3} + \ldots
\]

\[
+ \frac{n! - (n-1)! - k_1 \ldots (n-2)! - \ldots - k_{n-1}}{y^n} + O\left(\frac{1}{y^{n+1}}\right)
\]

for \( y \to \infty \). It follows that \( f(y) = 1 + O\left(\frac{1}{y^n}\right) \), i.e.

\[
\sum_{i=0}^{n} \frac{i!}{y^{i+1}} = \frac{1 + O\left(\frac{1}{y^n}\right)}{y - 1 - \sum_{i=1}^{n} \frac{k_i}{y^i}} = \frac{1}{y - 1 - \sum_{i=1}^{n} \frac{k_i}{y^i}} + O\left(\frac{1}{y^{n+2}}\right).
\]

We denote \( y = \log x \), and using the relations of type (1) it follows that

\[
\pi(x) = \frac{x}{\log x - 1 - \sum_{i=1}^{n} \frac{k_i}{\log^{i+1} x}} + O\left(\frac{x}{\log^{n+2} x}\right)
\]

(2)

Consider

\[
\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_n (1 + \alpha_n(x))}{\log^n x}}.
\]

Combining this formula with (2) yields \( k_n \alpha_n(x) = O\left(\frac{1}{\log x}\right) \), from which it follows that \( \lim_{x \to \infty} \alpha_n(x) = 0 \). \( \square \)

**Remark 2.** It can be shown immediately that \( k_1 = 1, k_2 = 3, k_3 = 13, k_4 = 71 \).
We give now a formula for $k_m$ (although not suitable for a direct computation).

**Theorem 3.** The coefficient $k_m$ is given by the relation:

$$k_m = \det \begin{pmatrix} m \cdot m! & 1! & 2! & \cdots & (m-1)! \\ (m-1)(m-1)! & 0! & 1! & \cdots & (m-2)! \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 \cdot 2! & 0 & 0 & \cdots & 1! \\ 1 \cdot 1! & 0 & 0 & \cdots & 0! \end{pmatrix}$$

Proof. The recurrence relations giving the coefficients $k_m$ are:

$$k_m + k_{m-1}1! + \cdots + k_1 (m-1)! = m \cdot m!$$

$$k_{m-1} + \cdots + k_1 (m-2)! = (m-1) \cdot (m-1)!$$

$$\cdots$$

$$k_2 + k_1 1! = 2 \cdot 2!$$

$$k_1 = 1 \cdot 1!$$

The determinant of this linear system is 1 and the result follows by Cramer’s rule.

As an application of the above results we are going to improve the following approximation, due to J.-M. de Koninck and A. Ivić, [1]:

$$\sum_{n=2}^{x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x + O(\log x).$$

Using Theorem 1 we are going to prove

**Theorem 4.**

$$\sum_{n=2}^{x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x - \log x - \log \log x + O(1).$$

Proof. It is enough to take

$$\pi(x) = \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{k(x)}{\log^2 x}},$$

where $\lim_{x \to \infty} k(x) = 3$, and it follows that

$$\frac{1}{\pi(n)} = \frac{\log n}{n} - \frac{1}{n} - \frac{1}{n \log n} - \frac{k(n)}{n \log^2 n},$$

for $n \geq 2$. Therefore we get that

$$\sum_{n=2}^{x} \frac{1}{\pi(n)} = \sum_{n=2}^{x} \frac{\log n}{n} - \sum_{n=2}^{x} \frac{1}{n} - \sum_{n=2}^{x} \frac{1}{n \log n} - \sum_{n=2}^{x} \frac{k(n)}{n \log^2 n}.$$  

For $x \geq e$, $f(x) = \frac{\log x}{x}$ is decreasing and thus

$$\frac{\log(k+1)}{k+1} \leq \int_{k}^{k+1} \frac{\log x}{x} dx \leq \frac{\log k}{k},$$

for $k \geq 3$. It follows immediately that

$$\sum_{n=3}^{x} \frac{\log n}{n} = \int_{3}^{x} \frac{\log t}{t} dt + O\left(\frac{\log x}{x}\right),$$

and so

$$\sum_{n=2}^{x} \frac{1}{n \log n} = \frac{1}{2} \log^2 x + O(1).$$

Similar arguments lead us to the relations

$$\sum_{n=2}^{x} \frac{1}{n} = \log x + O(1),$$

and

$$\sum_{n=2}^{x} \frac{1}{n \log n} = \log x + O(1).$$

As there exists $M > 0$ with $|k(x)| \leq M$, and $\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$ is convergent, it follows that $\sum_{n=2}^{x} \frac{k(n)}{n \log^2 n} = O(1)$, and the proof is complete.

**References**

