Problemen/UWC

Universitaire Wiskunde Competitie

The Universitaire Wiskunde Competitie (UWC) is a ladder competition for students. Others may participate 'hors concours'. The results can also be found on internet at: http://www.nieuwarchief.nl/uwc

Each issue contains three problems A, B and C. A total of 12 points can be obtained for each problem: 8 for a complete and correct answer, at most 2 points for elegance, and at most 2 points for possible generalisations. To compute the overall score, the totals for each problem are multiplied by a factor 3, 4 and 5, respectively.
The three best contributions will be honoured with a Sessions Prize of respectively 100, 50 and 25 Euro. The points of the winners will be added to their total after multiplication by a factor of respectively 0, 1/2 and 3/4. The highest ranked participant will be given a prize of 100 Euro, after which he/she starts over at the bottom of the ladder with 0 points.
Twice a year there is a Star Problem, of which the editors do not know any solution. Whoever first sends in a correct solution within one year will also receive a prize of 100 Euro.
Group contributions are welcome. Submission by email (in LaTeX) is preferred; participants should repeat their name, address, university and year of study at the beginning of each problem/solution. The submission deadline for this session is March 1, 2006.
The Universitaire Wiskunde Competitie is sponsored by Optiver Derivatives Trading, the Nederlandse Onderwijs Commissie voor Wiskunde and the Vereniging voor Studie- en Studentenbelangen in Delft.

Problem A (Unknown proposer)
We have $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$. Consequently partial sums must satisfy

$$\sum_{k\in K} \frac{1}{k(k+1)} < 1.$$

Show that for any $q \in \mathbf{Q}$ satisfying $0 < q < 1$, there exists a finite subset $K \subseteq \mathbf{N}$ so that

$$\sum_{k\in K} \frac{1}{k(k+1)} = q.$$

Problem B (Proposed by Matthijs Coster)
We consider the progressive arithmetic and geometric means of the function sequence $f_n(x) = x^{n-1}, n \in \mathbf{N}, x > 0, x \neq 1$. These are

$$A_n = A_n(x) = \frac{1}{n}(1 + x + x^2 + \cdots + x^{n-1}) = \frac{x^n - 1}{n(x-1)}$$

and

$$G_n = G_n(x) = (x^{1+2+\cdots+(n-1)})^{\frac{1}{n}} = x^{\frac{n-1}{2}}.$$

The *Martins-property* reads $A_{n+1}/A_n \geq G_{n+1}/G_n$. In our case this gives

$$\frac{n}{n+1}\,\frac{x^{n+1}-1}{x^n - 1} \geq \sqrt{x}.$$

Prove, more generally, that

$$\frac{a}{a+1}\,\frac{x^{a+1}-1}{x^a - 1} \geq \sqrt{x} \text{ for } a > -\frac{1}{2}, x > 0, x \neq 1.$$

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Problem C (Proposed by Roger Hendrickx and Rob van der Waall)

A *finite geometry* is a geometric system that has only a finite number of points. For an *affine geometry*, the axioms are as follows:

1. Given any two distinct points, there is exactly one line that includes both points.
2. The parallel postulate: Given a line L and a point P not on L, there exists exactly one line through P that is parallel to L.
3. There exists a set of four points, no three collinear.

We denote the set of points by $\mathbf{P}$, and the set of lines by $\mathbf{L}$. Let σ be an automorphism of $(\mathbf{P}, \mathbf{L})$ (meaning that three collinear points of $\mathbf{P}$ are mapped onto three collinear points of $\mathbf{P}$ and three noncollinear points of $\mathbf{P}$ are mapped onto three noncollinear points of $\mathbf{P}$). Prove that there exists a point $P \in \mathbf{P}$ with $\sigma(P) = P$ or a line $L \in \mathbf{L}$ with $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

Star Problem (Unknown proposer)

We have $\sum_{k=2}^{\infty} 1/k^2 = (\pi^2/6) - 1$. Consequently partial sums must satisfy

$$\sum_{k \in K} \frac{1}{k^2} < \frac{\pi^2}{6} - 1.$$

Given any $q \in \mathbf{Q}$ satisfying $0 < q < (\pi^2/6) - 1$, does there exist a finite subset $K \subseteq \mathbf{N} \setminus \{1\}$ so that

$$\sum_{k \in K} \frac{1}{k^2} = q?$$

Edition 2005/2

For Session 2005/2 of the Universitaire Wiskunde Competitie we received submissions from Herbert Beltman, Ruud Jeurissen, H. Reuvers, Piet Stam, and Jaap Spies.

Problem 2005/2-A A student association organises a large-scale dinner for 128 students. The chairs are numbered 1 through 128. The students are also assigned a number between 1 and 128. As the students come into the room one by one, they must sit at their assigned seat. However, one of the students is so drunk that he can't find his seat and takes an arbitrary one. Any sober student who comes in and finds his seat taken also takes an arbitrary one. The drunken student is one of the first 64 students. What is the probability that the last student gets to sit in the chair assigned to him?

Solution This problem was solved by Herbert Beltman, Ruud Jeurissen, Jaap Spies, and Piet Stam. The solution below is based on the solution of Herbert Beltman.

Let D be the seat of the drunken student and N be the seat of the last student. Before student D enters the room both chairs are empty. The drunken student, or one of the students after him, will sit in one of these seats. The last student will sit in the other one. The probability that he will sit in his own chair is $\frac{1}{2}$, and this probability is independent of the number of students and the time the drunken student entered.

Problem 2005/2-B We consider amino acids A, B and C and proteins formed by an ordered sequence of these. We also consider 9 enzymes which modify the proteins by replacing two adjacent amino acids by two other amino acids. These substitutions are given by:

$$\begin{aligned} &AA \to BC, \quad AB \to CC, \quad AC \to BA, \\ &BA \to CB, \quad BB \to CA, \quad BC \to AA, \\ &CA \to BB, \quad CB \to AC, \quad CC \to AB. \end{aligned}$$

We define classes of proteins as follows. If a protein has been modified by an enzyme, then it still belongs to the same class of proteins. Two proteins belong to two different classes of proteins if there doesn't exist a set of enzymes that is able to modify one of the proteins into the other.

1. How many classes of proteins consisting of 12 amino acids do there exist?
2. How many proteins belong to the class of proteins of $ABCCBAABCCBA$?

Solution This problem has been solved by Herbert Beltman and Ruud Jeurissen. The solution below is based on the solution of Herbert Beltman.

We rewrite the problem by substituting 0 for A, 1 for B, and 2 for C. A protein of length n can be written as $P = a_0a_1 \dots a_{n-1}$, where $a_k \in \{0,1,2\}$. The substitutions in the problem can now be written as:

$$\begin{aligned} &00 \to 12, \quad 01 \to 22, \quad 02 \to 10, \\ &10 \to 21, \quad 11 \to 20, \quad 12 \to 00, \\ &20 \to 11, \quad 21 \to 02, \quad 22 \to 01. \end{aligned}$$

We introduce a convenient transformation of the problem. Define

$$f(P) = b_0b_1 \dots b_{n-1},$$

where $b_k = a_k + k \bmod 3$. So again $b_k \in \{0,1,2\}$. We will call the new protein the *mutated* protein $Q = f(P)$. It is clear that f is a bijection.

Now we consider the transformations. The original transformations can be summarised by:

$$\begin{aligned} xx &\to (x+1)(x+2) \\ (x+1)(x+2) &\to (x+2)(x+2) \\ x(x+2) &\to (x+1)x \end{aligned}$$

where $x \in \{0,1,2\}$ and the numbers have to be taken modulo 3.

For the mutated proteins Q we get the transformations:

$$\begin{aligned} x(x+1) &\to (x+1)x \\ x(x+2) &\to (x+2)x \\ xx &\to (x+1)(x+1) \end{aligned}$$

We can summarize these transformations for the mutated proteins into two rules.

1. Swap two consecutive aminoacids if they differ.
2. If two consecutive amino acids are equal, increase both by one (modulo 3).

By applying Rule 1 numerous times to a mutated protein Q we can reach any permutation of Q. We can replace Rule 1 by:

1′. Replace Q by some permutation of Q.

Furthermore, we observe that by applying Rule 2 more often we can replace two consecutive equal amino acids by two other equal amino acids. We can replace Rule 2 by:

2′. Replace two consecutive equal amino acids by two other equal amino acids.

Only by applying Rule 2, the number of a given amino acid in Q can be changed. These changes are only applied to an even number of amino acids.

Let $Z(Q) = (z_0, z_1, z_2)$, where z_i is the number of amino acids i in Q modulo 2. Since $z_0 + z_1 + z_2 \equiv Z(Q) \bmod 2$, we have:

If the number of amino acids in Q is even then $Z(Q)$ can only be one of $(0,0,0), (1,1,0), (0,1,1), (1,0,1)$, while if the number of amino acids in Q is odd then $Z(Q)$ can only be one of $(0,0,1), (0,1,0), (1,0,0), (1,1,1)$.

It is easy to construct mutated proteins satisfying each of the eight cases (4 for the even case, 4 for the odd case). It is clear that these classes are all different (since only an even number of amino acids can be substituted for an even number of other amino acids). We even have the following lemma, which implies that the result to part 1 is 4.

Lemma. Two proteins Q_1 and Q_2 belong to the same class if and only if Q_1 and Q_2 have the same number of amino acids and $Z(Q_1) = Z(Q_2)$.

Proof. Suppose Q_1 and Q_2 are two proteins with the same number of amino acids, and with the number of amino acids 0, 1 and 2 in Q_1 equal to $z_0 + 2x_0$, $z_1 + 2x_1$, $z_2 + 2x_2$ respectively, and in Q_2 equal to $z_0 + 2y_0$, $z_1 + 2y_1$, $z_2 + 2y_2$.

We will show that these two proteins belong to the same protein class. We apply the following algorithm to both proteins Q_1 and Q_2:

- Sort Q_i in ascending order by applying Rule 1.
- Replace as many pairs 11 and 22 by 00 as possible and sort again.

For both proteins we obtain a protein Q with $2(x_0 + x_1 + x_2) + z_0$ zero's followed by z_1 ones and z_2 twos. (Notice that $x_0 + x_1 + x_2 = y_0 + y_1 + y_2$). ☐

Solution part 2

We have to calculate the exact size of a protein class. We define $g(n, z_0, z_1, z_2)$ as the number of proteins Q of size n satisfying $Z(Q) = (z_0, z_1, z_2)$. By symmetry arguments we have

$$g(n,1,1,0) = g(n,0,1,1) = g(n,1,0,1)$$

and

$$g(n,0,0,1) = g(n,0,1,0) = g(n,1,0,0).$$

Since the number of proteins of length n is 3^n we have

$$g(n,0,0,0) + g(n,1,1,0) + g(n,0,1,1) + g(n,1,0,1) = 3^n$$

for even n and

$$g(n,1,1,1) + g(n,0,0,1) + g(n,0,1,0) + g(n,1,0,0) = 3^n$$

for odd n. Given an arbitrary protein, we can adjoin an arbitrary amino acid and get a protein of length one more. As a formula this gives:

$$g(n,0,0,0) = g(n-1,0,0,1) + g(n-1,0,1,0) + g(n-1,1,0,0) = 3^{n-1} - g(n-1,1,1,1),$$

and

$$g(n,1,1,1) = g(n-1,0,1,1) + g(n-1,1,0,1) + g(n-1,1,1,0) = 3^{n-1} - g(n-1,0,0,0).$$

Therefore we find

$$g(n,0,0,0) = 3^{n-1} - g(n-1,1,1,1) = 3^{n-1} - 3^{n-2} + g(n-2,0,0,0).$$

Since $g(2,0,0,0) = 3$ and $g(2,0,1,1) = g(2,1,0,1) = g(2,1,1,0) = 2$ we find by induction that

$$g(n,0,0,0) = \frac{1}{4}(3^n + 3)$$

and

$$g(n,0,1,1) = g(n,1,0,1) = g(n,1,1,0) = \frac{1}{4}(3^n - 1).$$

We need to know whether the mutated protein related to the original protein ABCCBAABCCBA belongs to $Z(0,0,0)$ or to one of the three other classes. This can be done by calculating the related mutated protein. We find $P = 012210012210$ and $Q = 021222021222$. Consequently $Z(Q) = (0,0,0)$ and therefore the size of the class is $g(12,0,0,0) = \frac{1}{4}(3^{12} + 3) = 132861$.

Problem 2005/2-C In what follows, P stands for the set consisting of all odd prime numbers; M is the set consisting of all natural 2-powers $1, 2, 4, 8, 16, 32, \ldots$; T is the set consisting of all positive integers that can be written as a sum of at least three consecutive natural numbers.

1. Show that the set theoretic union of P,M, and T coincides with the set consisting of all natural numbers..
2. Show that the sets P,M, and T are pairwise disjoint.

3. Given $b \in T$, determine $t(b)$ in terms of the prime decomposition of b, where by definition $t(b)$ stands for the minimum of all those numbers $t > 2$ for which b admits an expression as sum of t consecutive natural numbers.
4. Consider the cardinality $C(b)$ of the set of all odd positive divisors of some element b of T. Now think of expressing this b in all possible ways as a sum of at least three consecutive natural numbers. Suppose this can be done in $S(b)$ ways. Determine the numerical connection between the numbers $C(b)$ and $S(b)$.

Solution This problem has been solved by Herbert Beltman, Ruud Jeurissen, H. Reuvers, Piet Stam, and Jaap Spies. The solution below is based on the solution of Jaap Spies.
Remark: In this problem we follow the convention not to include zero in the natural numbers.
First let

$$a = p_0^{e_0} \cdot p_1^{e_1} \cdots p_m^{e_m}$$

be the prime decomposition of a positive integer a with $p_0 = 2$, $e_0 \geq 0$ and $p_1, \ldots, p_m$ odd primes with $e_i > 0$ for $i = 1, \ldots, m$. We want to write a as the sum of k consecutive natural numbers starting with n.

$$a = n + (n+1) + \cdots (n+k-1) = k \cdot n + \frac{k(k-1)}{2} = k(2n+k-1)/2.$$

So $2a = k \cdot (2n+k-1)$. We define k to be the smallest factor, hence $k < \sqrt{2a}$. We observe that only one of the factors is odd.

Solution to parts 1 and 2
When a is a power of 2 we can only have $k = 1$. A power of two is clearly not an odd prime and vice versa. An odd prime can only be written as a sum of 2 consecutive natural numbers ($k = 2$). For all other positive integers we have at least one odd prime divisor p_i. Let $k = p_i \geq 3$ and $n = (2a/k - k + 1)/2$. It follows that a can be written as the sum of at least three consecutive positive integers starting with n. The remaining part of the proof is trivial.

Solution to part 3
Let $b = a \in T$ and p_1 be the smallest odd prime divisor of b. From (2) it follows that if $e_0 = 0$, meaning b is odd, we have $t(b) = p_1$, else $t(b) = min(2^{e_0+1}, p_1)$.

Solution to part 4
Let again $b = a \in T$. We use the prime decomposition (1) to find the number of odd divisors of b. We easily see that this number must be $(e_1+1) \cdot (e_2+1) \cdots (e_m+1)$. So $C(b) = (e_1+1) \cdot (e_2+1) \cdots (e_m+1)$. $S(b)$ is the number of ways b can be expressed as sum of at least three positive integers.
From (2) it follows that for each odd divisor of b we can find a $k < \sqrt{2a}$. We must exclude $k = 1$ and $k = 2$. Only in case of an odd b can we have $k = 2$, so $S(b) = C(b) - 2$ if b is odd and $S(b) = C(b) - 1$ if b is even.

Results of Session 2005/2

	Name	A	B	C	*Total*
1.	Herbert Beltam	8	10	8	104

Final Table after Session 2005/2
We give the top 3, the complete table can be found on the UWC website.

	Name	*Points*
1.	Peter Bruin	99
2.	Hendrik Hubrechts	90
3.	Syb Botma	42