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Brouwerlezing

Stochastic geometry and interacting fields

Op het 38ste Nederlands Mathematisch Congres op 4 april 2002 in Eindhoven werd aan Michael Aizenman de Brouwermedaille toegekend. Hij ontving deze onderscheiding voor zijn baanbrekende werk op het gebied van de mathematische fysica. Bij de uitreiking hield Aizenman onderstaande lezing.

I would like to express my joy and deep gratitude for being honored with the Brouwer Medal. The award comes from a most esteemed professional society, of colleagues amongst whom are the originators of various fundamental contributions to the area in which I have worked.

The award conveys a recognition not only of the recipient, but also of the field of mathematical physics. By its very name, this field has roots in more than one discipline. It invites a diversity of perspectives, whose combined expression can be found in mathematical results which advance our understanding of fundamental physics issues. The field has been enriched through the contributions of many colleagues, which has made working in it an exciting experience.

In this article I intend to outline some of the issues which have been tackled in one of the active areas of research in mathematical physics. I shall do so omitting most of the technicalities, though some of these were presented in the Brouwer lecture. Even before explaining more, let me note a curious link with the work of L.E.J. Brouwer. Some of the recent developments in the subject concern the nature of the continuum theory which emerges through the process of the *scaling limit*. As is explained below, this limiting procedure corresponds to taking down to zero the length scale at which the physics model is formulated with its detailed description of the fundamental variables and their dynamics. In the continuum theory which captures such a limit (for which the discrete models serve as a scaffold which is eventually taken down), each neighborhood represents a vast 'universe' in whose microscopic state there is only little continuity with 'adjacent' neighborhoods. Thinking along this line, one quickly reaches some perplexing issues which are related to the vastness of the continuum. L.E.J. Brouwer had well appreciated the challenges presented by the vastness of the continuum topology, and addressed it in some of his work. Apparently, we are still tantalized by the many related issues.

Criticality in Statistical Mechanics

The topic on which I shall focus in these comments concerns collective phenomena in systems with many degrees of freedom, as encountered in statistical mechanics, in particular, issues related to critical behavior. Here, criticality of the system is synonymous with an enhanced capacity of response to small changes. These may be induced in a number of ways: through the varying of control parameters, through small changes in the 'typical' configuration, or through naturally occurring fluctuations. Criticality is often encountered at phase transitions, where the state of the system undergoes a non-analytic change as a function of the external control parameters, though it has been noted that there are abundant mechanisms for 'self-organized criticality' as well (P. Bak).

It is an often encountered situation in physics that there is a very considerable difference between the length scale on which various effects of interest may be observed and the scale of distances at which the underlying interactions occur. In such a case it is natural to formulate a continuum theory for the phenomena observed on the larger scale, which may be quite different from the smaller scale appropriate for the detailed description of the underlying model. The notable fact is that one often encounters laws which make sense and are valid on the larger scale with no reference to the smaller scale. These then ought to be mathematically derivable from the more fundamental small-scale description through the process of a scaling limit, the limit in which the smaller scale is taken down to zero (relative to the scale of the observed phenomena).

Statistical mechanical systems are typically described by large arrays of correlated random variables, which are indexed by either a discrete lattice (e.g., $\mathbf{Z}^d$) or a continuum space (e.g., $\mathbf{R}^d$) and are correlated through interactions which are 'local' in the natu-

Gibbs measure
A Gibbs measure for a finite-volume system is a probability distribution with a density given by the 'canonical ensemble' prescription of the American physicist Josiah Willard Gibbs, of the form $exp[-(\beta \times energy)]$, with β denoting the inverse of the temperature. The generalization to infinite-volume systems requires the use of conditional probability densities for finite volumes embedded in infinite volumes, expressible in this Gibbsian form.

ral sense. (The case where the space is the continuum of course requires some extra care, as does the extension to quantum-mechanical situations where the variables are replaced by non-commuting operators.)

The earliest contribution of statistical mechanics has been to reconcile the thermodynamic behavior observed in *macroscopic* physical systems with their more basic *microscopic* descriptions, in particular, to shed light on *entropy* as a derived concept (L. Boltzmann, J.W. Gibbs). A more recent challenge has been to understand critical phenomena. As mentioned above, critical systems are encountered at the onset of various phase transitions (gas to liquid, paramagnetic to ferromagnetic, etc.). Mathematically, this is manifest by the random variables having anomalously large fluctuations, i.e., divergent on the scale of the natural 'central limit' fluctuations.

Interest in this topic has been fed by physicists' observation that various features of the critical behavior in physics are shared by systems which are quite different on the microscopic level. This 'universality' has suggested that the critical behavior may be guided by the relations among some aggregate quantities. This in turn has led to the notions (firmly embraced, even if not rigorous) of 'scaling laws', 'universality classes', and 'renormalization group' (B. Widom, L. Kadanoff, M.E. Fisher, K.G. Wilson). Since the models are typically not amenable to exact solutions, it has been of interest to develop mathematical results which can shed light both on the qualitative features of some specific instructive systems and on the general structure which emerges on the larger scale. One may regard as the next layer of research in equilibrium statistical mechanics the growing collection of works which address issues concerning the *scaling limits of critical systems*. More will be said on this below.

Geometrization of Correlations

A simply stated example which already exhibits interesting behavior, including phase transitions and related critical phenomena, is provided by the so-called Potts model of ferromagnetism (the latter being an example of a cooperative phenomenon). The system is described by the collection of random variables $\sigma = \{\sigma_x\}_{x\in\mathbf{Z}^d}$ ('attached' to the d-dimensional lattice $\mathbf{Z}^d$), each of which takes Q distinct values, with a joint probability distribution of the form of a 'Gibbs measure'

$$\frac{1}{\text{Norm.}}\ \exp\{-\beta H(\sigma)\}\ \rho(d\sigma),$$

where $\rho(d\sigma)$ is a product measure which gives equal weights to all the configurations in a finite system, $H(\sigma)$ is the 'Hamiltonian' which gives the agreement-enhancing (ferromagnetic) interaction

$$H(\sigma) = -\sum_{u,v\in\mathbf{Z}^d} J_{|u-v|}\,\delta_{\sigma_u,\sigma_v},$$

with $\{J_{|u-v|}\}$ rapidly decaying in $|u-v|$ and non-negative, and β a control parameter which in physical terms corresponds to the 'inverse temperature'. In dimensions $d > 1$, the (nearest neighbor) Potts model exhibits phase transitions from the disordered phase (at $\beta < \beta_c$, with β_c the critical inverse temperature), where the infinite-volume limit of the finite-volume Gibbs measure is ergodic under the group of translations, to a mixture of ordered phases (at $\beta > \beta_c$), where the latter limit is a superposition of Q distinct ergodic measures, each exhibiting a higher density for one of the Q possible values over the other values. The coexistence of different Gibbs measures with different densities represents a 'first-order phase transition', at which some local quantities have discontinuous expectation values as a function of thermodynamic parameters such as chemical potential, temperature, etc. Critical behavior in the sense usually ascribed to it (i.e., enhanced sensitivity to small changes, as described above, but less drastic than plain discontinuity) is associated with a 'second-order phase transition', in the classification of P. Ehrenfest. Depending on Q and the dimension, the Potts model may exhibit critical behavior at the particular value β_c at which the discontinuity sets in. Indeed, this is proven to be the case for all the Potts models with $Q = 2$ (the so-called Ising models) with short-range translation-invariant ferromagnetic interactions in dimensions $d > 1$.

In non-critical models the fluctuations of local variables have only short-range correlations, which decay exponentially with a finite 'correlation length'. Critical systems are characterized (indeed, defined) by the divergence of that correlation length. At critical points the systems exhibit large scale fluctuations, expressed through either the correlation functions or some suitable characteristics of 'typical configurations', e.g., the sum

$$\sum_{u\in[-L,L]^d\cap\mathbf{Z}^d} [\sigma_u - \mathbf{E}(\sigma_u)]$$

being of order L^λ with λ exceeding the central limit value $\lambda = d/2$. This λ is an example of an exponent which is expected to show a remarkable degree of independence of many of the local parameters of the model, for instance, the values of the coupling constants $\{J_{|u-v|}\}$ within the class of short-range d-dimensional translation-invariant ferromagnetic interactions. Such 'universality' is expected of various other so-called 'critical exponents'. (This notion extends the original observation of the experimentally discovered universality of the 'scaling laws' and 'scaling expo-

Potts model
A Potts variable (or Potts spin) is a q-valued object $\sigma = 1,\ldots,q$. In the ferromagnetic Potts model, the energy of a configuration in some finite volume is given by the sum over all nearest-neighbor pairs (i,j) in that volume of the functions $-\delta(\sigma_i,\sigma_j)$. This implies that the lowest energy is reached when at all sites one chooses the same value (out of the q possible ones).

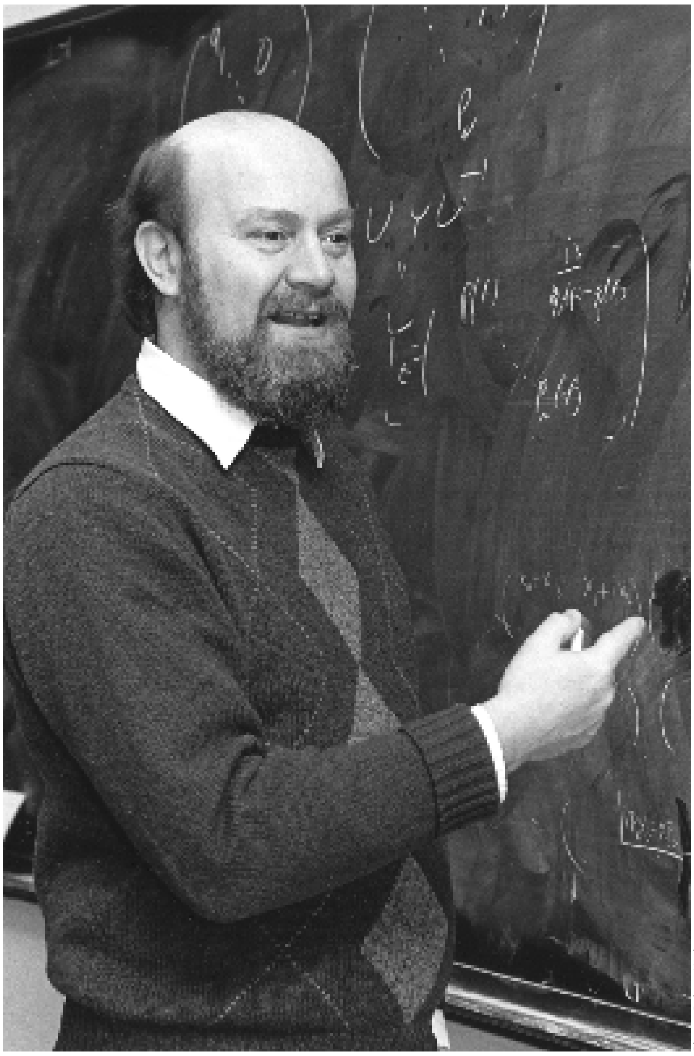

Photo: Robert Mathews

nents' of the related thermodynamics.)

Among the very helpful observations for the study of such systems has been the realization of C.M. Fortuin and P.W. Kasteleyn that the correlations among the random variables with the above probability distribution can be accounted for through the geometric connectivity properties of an associated system of *random clusters* (which since has been extended to various other systems). For a simple example, one may note via the relation

$$e^{\beta(\delta-1)} = e^{-\beta} + (1-e^{-\beta})\delta, \qquad \delta \in \{0,1\},$$

that for an isolated pair of Potts variables $\{\sigma_u, \sigma_v\}$ the corresponding probability distribution admits a 'polarized' representation, in which there is an added 'random bond' binary variable $n_{u,v}$, conditioned on which $\{\sigma_u, \sigma_v\}$ are constrained to be either equal, if $n_{u,v} = 1$, or completely independent, if $n_{u,v} = 0$. For the more extended system the enhanced description involves a system of random bond variables $\{n_{\{u,v\}}\}$, whose configurations yield random partitions of the lattice into clusters. Conditioned on the auxiliary bond variables, the Potts variables take values which are constrained to be constant on each of the connected clusters, and statistically independent between different clusters.

This random cluster representation allows to express the correlation between a pair of variables $\{\sigma_u, \sigma_v\}$, which usually is given by only a subtle difference between two probabilities, as the probability (up to a simple multiplicative constant) that in the random configuration of the associated system of random bond variables the two sites, $u, v \in \mathbf{Z}^d$, are connected by a path of occupied bonds. The existence of more than one ergodic Gibbs measure for the Potts model corresponds to the occurrence of an infinite cluster of these random bonds. This provides an example of a percolation transition. This approach also permits to present various characteristic exponents which are associated with the decay of correlations in the critical case, like η in

$$|\mathbf{E}\,(\delta_{\sigma_u,\sigma_v}) \;-\; Q^{-1}| \approx \text{Const.}/|u-v|^{d-2+\eta}, \qquad |u-v| \to \infty,$$

as related to dimension-like parameters of random clusters, which in the scaling limit appear to form random fractal objects.

Implications for the Critical Behavior in High and in Low Dimensions

The above geometrization of critical phenomena is reminiscent of the picture advocated by B. Mandelbrot. From a technical perspective, it presents two key advantages:

i. it brings positivity into the description of subtle cancellations (e.g., as seen already at the level of two-point correlations);
ii. it permits to account for multi-correlations in terms of intersection properties of random clusters.

Positivity is a potent tool; it has been employed both for generally useful inequalities (C.M. Fortuin - P.W. Kasteleyn - J. Ginibre, J. van den Berg - H. Kesten, J. van den Berg - C. Maes), and for specific results concerning the phase transitions in models of interest (works by M. Aizenman, J. Chayes, L. Chayes, R. Fernández, J. Fröhlich, C.M. Newman).

The reduction of multicorrelations to the intersection properties of random clusters adds a useful perspective. Loosely speaking: if these clusters tend to have specific dimensions (as, for instance, does the path of Brownian motion, which appears in the scaling limit of random chains with no repulsion and which are two-dimensional in any $d \geq 2$), then their intersections would be very rare in sufficiently high dimensions ($d > 4$), but not that unusual in low dimensions (certainly in $d = 2$ dimensions, and, as it turns out, also in $d = 3$). Such observations, of course with more precision and much more analytical input than what is described here, have been instrumental for rigorous derivations of the fact that in various systems the scaling exponents take simple values in sufficiently high dimensions (e.g., $d > 4$ for Ising models, $d > 6$ for certain percolation models, $d > 4$ for certain self-repellent random walk models). Rigorous results along this vein (e.g., [1–2]) include contributions by M. Aizenman, D. Brydges, R. Fernández - J. Fröhlich - A. Sokal, T. Hara, R. van der Hofstad, F. den Hollander, C.M. Newman, G. Slade, T. Spencer, and others.

The Scaling Limit

Since the fluctuations observed in critical systems give rise to features observed on arbitrary scales, it is natural to consider them through the limit in which the scale of the underlying microscopic description is taken down to zero. This is the scaling limit in statistical mechanics. Many of the underlying variables which are part of the explicit formulation of the model have to drop out

Percolation
In percolation theory one consider networks of sites and of bonds between them, which can be open with probability $p \in (0, 1)$ and closed with probability $1 - p$, independently of each other. The main interest is to study the geometric properties of the 'clusters' of sites and/or bonds that are all open. At a critical value $p_c \in (0, 1)$, the system changes its behavior drastically: for $p < p_c$ all clusters are finite, while for $p > p_c$ there is one infinite cluster that pervades the whole system.

in this limit (essentially, for reasons of measurability). One may, nevertheless, seek a mathematical theory with a collection of variables which express those fluctuations extending in space over scales close to the one on which the limit is focused. Such a theory may also be rich in mathematical structure: asymptotic symmetries, such as emergent rotational invariance or conformal invariance, would become exact laws. The theory should also capture a *rich local structure*. Its description requires tools which go considerably beyond smooth functions, since in the scaling limit even infinitesimal neighborhoods contain an infinite amount of detail.

Continuum theories with the above characteristics have attracted physicists' attention also in relation to quantum physics, where the scaling limit is essential for a constructive grasp of the meaning of quantum field theory. It turns out that many of the mathematical issues involved in statistical mechanics and in quantum field theory are related, and form bridges between the two subjects.

One may note that the challenge of understanding the continuum appears to include two different, but not unrelated, tasks. One is to develop a mathematical language with terms suitable to the continuum limit, dealing with the infinite (though nevertheless somewhat limited) collection of quantities which are meaningful on that scale. The continuum symmetries should also find a simple expression. This challenge is taken by 'field theory' [3], which carries its own terminology, the tool chest including notions like fields, operators, and stress-energy tensor (to mention but some of the contributors: A.A. Belavin - A.M. Polyakov - A.B. Zamolodchikov, C.G. Callan, L. Faddeev, G. 't Hooft, L. Kadanoff, K. Symanzik, K.G. Wilson). Lest one gets the feeling that this chapter of the book may be closed, let me add that right now remarkable new tools, applicable to a variety of two-dimensional systems, are being added through the SLE_κ processes introduced recently by O. Schramm [6] (the so-called 'Stochastic Loewner Evolutions', parametrized by κ).

The other task in understanding the physical continuum is to start from analytically tangible models, either discrete in nature or continuous but with 'cutoffs' which render the relevant integrals absolutely convergent, and to study their behavior in the suitable limit. Such a program has been the goal of 'constructive field theory', an area in which various rigorous results were derived, both constructive — in particular, in dimensions $d = 2, 3$ (J. Glimm - A. Jaffe, D. Brydges, J. Fröhlich, A. Sokal, T. Spencer) — and some outlining the limitations of possible approaches in high dimensions (M. Aizenman [1], J. Fröhlich). I shall not comment here on this very extensive body of work. However, let me add that new developments are now taking place in the area of two-dimensional systems, which may shed further light and a new perspective on conformal field theory in two dimensions (G. Lawler - O. Schramm - W. Werner [7], R. Kenyon, S. Smirnov [8]). Impetus for the later work was provided, to some extent, by the desire for a better understanding of scaling limits in stochastic geometric models such as those mentioned above [4], [5].

Percolation

The representation of a phase transition as a percolation phenomenon in an interacting system of random bonds, as described above, has further boosted the motivation for the study of the percolation transition in models which are simpler to present, such as systems of independently placed geometric objects.

Specific formulations of percolation models include the bond percolation model, where bonds (nearest-neighbor links) of a lattice are 'open' with probability p and 'closed' with probability $1 - p$, independently of each other. Alternatively, the model may be based on randomly occupied lattice sites, or randomly placed disks in the continuum. Such models have provided fertile grounds, and there has been plenty of positive feedback between the mathematical analysis of percolation systems and of systems of interacting 'spin' variables (J. Hammersley, H. Kesten, M. Aizenman, D. Barsky, T. Hara, G. Slade, and more recently, R. Langlands, Y. Saint-Aubin, J. Cardy, G. Lawler, O. Schramm, W. Werner, S. Smirnov).

As with other systems of statistical mechanics, a very detailed structure has emerged in two dimensions. Many insights were obtained by methods related to physicists' analysis of two-dimensional Potts models (B. Nienhuis, M. den Nijs, B. Duplantier, J. Cardy). Some key results of a qualitative nature were derived rigorously by other methods (L. Russo, P. Seymour, D. Welsh, H. Kesten). Recently, the subject was revisited with the goal of shedding further light on the possible formulations of the scaling limit, in two as well as in higher dimensions.

The simplest example of a stochastic geometric model with a well-understood scaling limit is provided by the simple random walk. Its scaling limit is the process of Brownian motion, which for this purpose is viewed through the Wiener measure on the

Brownian motion
Brownian motion is a random path in Euclidean space whose increments are 'completely random'. This means that the changes in the path over disjoint time intervals are independent from each other, resulting in a 'fully noisy motion'.
Brownian motion is called after the botanist Robert Brown, who observed the motion of pollen particles early in the 19th century. The first theoretical description is due to Einstein, in the most-cited paper of his miracle year 1905. The further development of the mathematical theory owes much to Norbert Wiener.
Brownian motion is used to model a large variety of random phenomena. As such it is a basic building block for probabilists. Random walk is a discrete space-time version of Brownian motion, where the increments take values in a space grid (or lattice) and run over a time grid. In the limit as the space-time grid tends to zero, random walk converges to Brownian motion.

space of continuous functions. This example exhibits a number of the characteristics which were mentioned above:

i. Details about the individual steps are lost in the limit, nevertheless, aggregate fluctuations of a suitable scale are part of the continuum description.
ii. The paths exhibit fluctuations on all scales, are rough (nowhere differentiable, 'tortuous' in a specific sense), yet retain Hölder-type continuity.
iii. Higher symmetry emerges in the scaling limit: the probability distribution is invariant under rotations, and in two dimensions is conformally invariant, up to time reparametrization.
iv. Various related quantities obey simple differential equations (e.g., hitting probabilities are harmonic functions). It would be of interest to extend this success to other stochastic geometric models. In particular, the Fortuin-Kasteleyn construction suggests that progress on this may shed further light on certain field theories.

Interesting questions are encountered when one asks which variables ought to be used for the scaling limit of critical percolation models. In joint work with A. Burchard, we present conditions under which the scaling limit may be expressed through a family of connected paths which are tortuous yet are Hölder continuous. However, the effectiveness of such a description would of course depend on the properties of the limiting object, and thus be dimension-dependent (a more detailed discussion is provided in ref. [5]).

In the special case of two dimensions the boundaries of connected clusters are curves. This fact has been used in two ways. In one approach, considerable insight has been obtained by approaching these curves as the level sets of random functions and by proceeding with some very insightful conjectures about the limiting distribution of such 'height functions' (B. Nienhuis, M. den Nijs). Another approach, departing from this observation, was proposed by O. Schramm, who has pointed out that there are severe limitations on the possible laws of random curves in two dimensions, if they are to possess the conformal invariance properties which correspond to existing conjectures about the scaling limit. This has led him to the introduction of the aforementioned SLE_κ processes, which are now the basis for a fast growing body of beautiful results (G. Lawler, O. Schramm, W. Werner, and others). There has also been progress in establishing links between specific models and continuum theories which have been conjectured to describe their scaling limits (R. Kenyon, S. Smirnov).

Our understanding of the situation in the interesting dimensions $d = 3, 4$ is more limited. However, the fog lifts as one moves to higher dimensions, where the large critical clusters behave as continuum objects which have enough space to 'unfold' without significant intersections (M. Aizenman, D. Barsky, T. Hara, G. Slade, R. van der Hofstad, F. den Hollander). In particular, T. Hara and G. Slade [9] have shown that the individual clusters attain the distribution of a critically and continuously branching Brownian motion ('integrated super-Brownian excursion').

In closing, let me say that I hope to have conveyed here some of the intertwined issues which are part of the fabric of modern statistical mechanics. One may note that both the subject discussed above, as well as other related areas, such as the theory of Gibbs states (see A. van Enter, C. Maes and S. Shlosman [10]), and the vast field of non-equilibrium statistical mechanics, have much benefited from excellent contributions by Dutch physicists and mathematicians. It has therefore been a special joy for me to have been honored by the invitation of *het Wiskundig Genootschap* to give the Brouwer lecture.

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It is not practical to give here proper references to the many works which were mentioned. Following are some pointers, with emphasis on the latter part of the discussion.

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