

De Universitaire Wiskunde Competitie (UWC) is een ladderwedstrijd voor studenten, georganiseerd in samenwerking met de Vlaamse Wiskunde Olympiade. De opgaven worden tevens gepubliceerd op de internetpagina <http://academics.its.tudelft.nl/uwc>

Ieder nummer bevat de ladderopgaven A, B, en C waarvoor respectievelijk 30, 40 en 50 punten kunnen worden behaald. Daarnaast zijn er respectievelijk 6, 8 en 10 extra punten te winnen voor elegantie en generalisatie. Er worden drie editieprijszen toegekend, van 100, 50, en 25 Euro. De puntentotalen van winnaars tellen voor 0, 50, en 75 procent mee in de laddercompetitie. De aanvoerder van de ladder ontvangt een prijs van 100 Euro en begint daarna weer onderaan. Daarnaast wordt twee maal per jaar een ster-opgave aangeboden waarvan de redactie geen oplossing bekend is. Voor de eerst ontvangen correcte oplossing van deze ster-opgave wordt eveneens 100 Euro toegekend.

Groepsinzendingen zijn toegestaan. Elektronische inzending in L^AT_EX wordt op prijs gesteld. De inzendtermijn voor de oplossingen sluit op 1 februari 2004. Voor een ster-opgave geldt een inzendtermijn van een jaar.

De Universitaire Wiskunde Competitie wordt gesponsord door Optiver Derivatives Trading en wordt tevens ondersteund door bijdragen van de Nederlandse Onderwijs Commissie voor Wiskunde en de Vereniging voor Studie- en Studentenbelangen te Delft.



Opgave A

For each non-negative integer n , let a_n be the number of digits in the decimal expansion of 2^n that are at least 5. For example, $a_{16} = 4$ since $2^{16} = 65536$ has four digits that are 5 or higher. Evaluate the sum $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$.

Opgave B

Let G be a group such that squares commute and cubes commute, i.e., $g^2h^2 = h^2g^2$ and $g^3h^3 = h^3g^3$ for all $g, h \in G$. Show that G is abelian.

Opgave C

Let $(X_n)_{n \geq 1}$ be a sequence of independent and identically distributed random variables with $P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}$. Set $S_n = \sum_{k=1}^n X_k$. Calculate

$$P\{\exists n \geq 1 \text{ such that } S_{3n} = n\}.$$

Editie 2003/2

Op de ronde 2003/2 van de Universitaire Wiskunde Competitie ontvingen we inzendingen van Filip de Smet en van Maarten Dobbelaar.

Opgave 2003/2-A

For $n \geq 1$ let r_n be the sequence

$$r_n = 3^n + 5, \quad n = 1, 2, \dots$$

Prove that for every $k \geq 1$ there exists an $n \geq 1$ such that 2^k divides r_n .

Oplissing Solutions by Maarten Dobbelaar, Bert Jagers, Filip de Smet, Benne de Weger. The solution of Jagers is as follows. First observe that

$$3^{2^{k-2}} \equiv 1 + 2^k \pmod{2^{k+1}} \quad (1)$$

for $k \geq 3$. We inductively show that for each k there exists an n such that $3^n \equiv -5 \pmod{2^k}$. Suppose that for $k \geq 3$ there exists an n such that $3^n \equiv -5 \pmod{2^k}$. Then $\pmod{2^{k+1}}$, either $3^n \equiv -5$, or $3^n \equiv -5 + 2^k$. In the first case we are done. In the second case, multiply by the congruence of (1) to find that $3^{n+2^{k-2}} \equiv -5 \pmod{2^{k+1}}$.

Benne de Weger uses 2-adic numbers and the 2-adic logarithm \log_2 , which has the property that $\text{ord}_2(\log_2(1+x)) = \text{ord}_2(x)$ if x is a 2-adic integer. Then $\text{ord}_2(3^n + 5) = \text{ord}_2(-1 + (-3)^{-n}5) = \text{ord}_2(\log_2((-3)^{-n}5)) = \text{ord}_2(\log_2(5) - n\log_2(-3))$. Write $\nu = \frac{\log_2(5)}{\log_2(-3)}$ to find that this order is equal to $\text{ord}_2(\nu - n) + \text{ord}_2(\log_2(-3)) = \text{ord}_2(\nu - n) + 2$. Now one can approximate ν by integers n to find $3^n + 5$ of arbitrary order.

Opgave 2003/2-B Prove that the function

$$f(x) = \sqrt{1+x} \cdot \ln \left(1 + \frac{1 + \sqrt{1+2x}}{1+x} \right)$$

is increasing on the interval $[0, \infty)$.

Opløsing By Filip de Smet and Bert Jagers. Both solutions have been assisted by Maple. Here is Filip de Smet's solution. Compute the derivative

$$f'(x) = \frac{1}{\sqrt{1+x}} \left(\frac{1}{2} \ln \left(1 + \frac{1 + \sqrt{1+2x}}{1+x} \right) - \frac{1}{\sqrt{1+2x}} \frac{x + \sqrt{1+2x}}{2+x + \sqrt{1+2x}} \right)$$

In order to prove that $f'(x) > 0$ we consider $g(x) = \sqrt{1+x} f'(x)$ and put

$$y = \sqrt{1+2x}, \quad \text{or} \quad x = \frac{y^2 - 1}{2} \quad \text{for } y \in [1, \infty)$$

This gives the expression

$$g(y) = \frac{1}{2} \ln \left(\frac{y^2 + 2y + 3}{y^2 + 1} \right) - \frac{1}{y} \frac{y^2 + 2y - 1}{y^2 + 2y + 3}$$

Since

$$\lim_{y \rightarrow \infty} g(y) = 0$$

and

$$g'(y) = -\frac{7y^4 + 4y^3 + 2y^2 + 4y + 3}{y^2(3 + y^2 + 2y)^2(1 + y^2)} < 0 \quad \forall y \in [1, \infty)$$

we find that $g(y) > 0$ and so $f(x)$ is increasing on $[0, \infty)$.

Opgave 2003/2-C

For which $n \in \mathbf{N}$ does there exist an enumeration

$$q_1, q_2, q_3, \dots$$

of \mathbf{Q}^n , such that $\|q_{k+1} - q_k\| = 1$ for all k .

Opløsing The problem was proposed by Michiel de Bondt. We present a solution offered by Jos Brakenhoff. The answer is $n \geq 5$. It is impossible to enumerate \mathbf{Q}^4 since for a unit length rational point $\left(\frac{a}{e}, \frac{b}{e}, \frac{c}{e}, \frac{d}{e}\right)$ one has

$$v_2(e) \leq v_2(a + b + c + d)$$

with $v_2(n)$ the number of factors 2 in n , for $n \in \mathbf{N}$. If $v_2(e) = 0$, then this is trivial. If $v_2(e) = 1$, then $a \equiv b \equiv c \equiv d \pmod{2}$ and $2|(a + b + c + d)$. If $v_2(e) > 1$ then there is no solution with a, b, c or d odd, so we can divide a, b, c, d and e by 2.

Furthermore, if the vectors $\left(\frac{a}{e}, \frac{b}{e}, \frac{c}{e}, \frac{d}{e}\right), \left(\frac{a'}{e'}, \frac{b'}{e'}, \frac{c'}{e'}, \frac{d'}{e'}\right) \in \mathbf{Q}^4$ satisfy

$$v_2(e) \leq v_2(a + b + c + d) \quad \text{and} \quad v_2(e') \leq v_2(a' + b' + c' + d'),$$

Oplossingen

then the sum

$$\left(\frac{a'e + ae'}{ee'}, \frac{b'e + be'}{ee'}, \frac{c'e + ce'}{ee'}, \frac{d'e + de'}{ee'} \right)$$

has the same property:

$$v_2(ee') \leq v_2(a'e + ae' + b'e + be' + c'e + ce' + d'e + de').$$

So, for example, $(\frac{1}{2}, 0, 0, 0)$ cannot be reached. This also shows that $(\frac{1}{2}, 0, \dots, 0)$ can not be reached in \mathbf{Q}^n with $n = 1, 2, 3$.

We prove that \mathbf{Q}^5 can be enumerated. Observe that the numbers that can be connected to the origin form a subgroup. First we show that for any prime power p^k the origin can be connected to $(\frac{1}{p^k}, 0, 0, 0, 0)$. Represent $(2p^k)^2 - 1$ as a sum of four squares

$$a^2 + b^2 + c^2 + d^2 = (2p^k)^2 - 1.$$

Then

$$\left(\frac{1}{2p^k}, \frac{a}{2p^k}, \frac{b}{2p^k}, \frac{c}{2p^k}, \frac{d}{2p^k} \right) + \left(\frac{1}{2p^k}, -\frac{a}{2p^k}, -\frac{b}{2p^k}, -\frac{c}{2p^k}, -\frac{d}{2p^k} \right) = \left(\frac{1}{p^k}, 0, 0, 0, 0 \right).$$

This shows that the whole set \mathbf{Q}^5 can be connected by segments of length 1. Since there are infinitely many unit-length-vectors in \mathbf{Q}^5 , any two points in \mathbf{Q}^5 can be connected by infinitely many disjoint paths. So \mathbf{Q}^5 can be enumerated by unit steps. The same argument holds for $n > 5$. So \mathbf{Q}^n can be enumerated by unit steps as well.

Uitslag Editie 2003/2

De weging van de opgaven is 3 : 4 : 5.

Naam	A	B	C	Totaal
1. Filip De Smet (Gent)	8	8	6	86
2. Maarten Dobbelaar	4	-	-	12

Ladderstand Universitaire Wiskunde Competitie

We vermelden alleen de top 5. Voor de complete ladderstand verwijzen we naar de UWC-website.

Naam	Punten
1. Mark Veraar	208
2. Team Filip Cools	199
3. Tom Claeys	138
4. Team Gerben Stavenga	136
5. Peter Bruin	99