Problemen

| Problem Section

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

Editor:
R.J. Fokkink
Technische Universiteit Delft
Faculteit Wiskunde
P.O. Box 5031
2600 GA Delft
The Netherlands
r.j.fokkink@its.tudelft.nl

Problem 35 (Floor van Lamoen)
Let two congruent circles (O_1) and (O_2) be tangent to each other, to segment AB and to the circumcircle of triangle ABC, in such a way that O_1, O_2 and C lie on the same side of AB. Prove that the point of contact of (O_1) and (O_2) is concyclic with A, B and the incenter I of the triangle ABC.

Problem 36
Let k be a positive integer, and write $n = 4k + 1$. Prove that $(n^n - 1)/(n - 1)$ is not a prime number.

Problem 37 (Open problem)
Does there exist a continuous surjection $f\colon [0,1] \to [0,1]^2$ such that every convex set has a convex image?

Solutions to Problem 26 and to volume 3, number 1 (March 2002)

Problem 26 (F. Roos)
Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

Solution This problem was solved by Frits Beukers, Jaap Spies and Christiaan van de Woestijne. They show that there are no such triangles. Jaap Spies gives several solutions. All solutions reduce the problem to finding rational points on an elliptic curve. Spies and Van de Woestijne use Heron's formula, but there is a more direct way to find the elliptic curve, as given by Beukers and by Spies, once again. Let a, b, c be the integer sides of the triangle, which we assume positive. Suppose the triangular area equals a^2. Then, using a as base, the height of the triangle should be $2a$. The perpendicular line on the base divides the base into two pieces of length x (which may be negative) and $a - x$. Then we should have

$$x^2 + (2a)^2 = b^2$$
$$(a-x)^2 + (2a)^2 = c^2$$

A subtraction shows that x is rational. Now denote $\xi = x/2a$, $\beta = b/2a$, $\gamma = c/2a$, which are all rational numbers. After division by $(2a)^2$ we find

$$\xi^2 + 1 = \beta^2$$
$$(1/2 - \xi)^2 + 1 = \gamma^2$$

It is a well-known fact that a rational number r has the property that $1 + r^2$ is a rational square if and only if $r = (s - 1/s)/2$ for some rational number s. Hence there exist rational numbers s, t such that $\xi = (s - 1/s)/2$ and $1/2 - \xi = (t - 1/t)/2$. Elimination of ξ gives

$$s + t - \frac{1}{s} - \frac{1}{t} = 1.$$

Clear denominators and we find $s^2t + st^2 - s - t - st = 0$. This is the equation of a non-singular cubic curve, which has a trivial rational point $s = t = 0$, i.e., an elliptic curve. Rational points on an elliptic curve are known to form an abelian group which can usually be computed. Introduce new coordinates x, y by

$$y = \left(\frac{s}{t} + 1\right)\frac{1}{t}, \qquad x = \frac{s}{t} + 1.$$

We then find the equation

$$y^2 + xy - y = x^3 - x^2.$$

Oplossingen | Solutions

This is a curve of conductor 17 which has four rational points. Beukers and Spies refer to the Cremona table of elliptic curves, which is available at William Stein's homepage http://modular.fas.harvard.edu/Tables/index.html. The rational points are $(0,0), (0,1), (1,0)$ and a point at infinity. These all give degenerate triangles. Van de Woestijne actually computes the rational points by hand, referring here and there to Silverman's book. Jaap Spies mentions that he spent a pleasant Christmas holiday reading J.S. Milne's excellent classroom notes, available at http://www.jmilne.org/math/.

Problem 27 (L. Bleijenga)
Let L be a Latin square of order n. Show that any matrix $A \subset L$ of order $a \times b$ with $a + b = n + 1$ contains all elements $1, 2, \ldots, n$.

Solution This problem received no solution. This is the solution by L. Bleijenga. After permutation one may assume that $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. If a symbol k does not occur in A, then every row of B has a k and so has every column of C. Then k occurs $a + b > n$ times, contradicting that L is a Latin square.

Problem 28 (H. van den Berg)
For integers k, m, n and a prime number $p \geq 5$ show that if $(k^2 - mn)^p + (m^2 - kn)^p + (n^2 - km)^p = 0$, then p divides all three numbers $k^2 - mn, m^2 - kn, n^2 - km$.

Solution This problem received no solution. Problem 28 can be solved by adapting the solution of Problem 7 that appeared in the first issue of volume 2 of Nieuw Archief.

Problem 29 (Lute Kamstra)
Let $n \in \mathbf{N}$, $h \in \mathbf{N}_0$ and let A be a subset of $\{1, 2, \ldots, n + h\}$ of size n. Count the number of bijective maps $\pi : \{1, 2, \ldots, n\} \to A$ such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$.

Solution Jaap van Oosten solved this open problem. The answer is 1, independent of A, n and h. Observe that this is trivially so if $n = 1$, or if $h = 0$. Write $A = \{1, \ldots, n + h\} - \{a_1, \ldots, a_h\}$ with $a_1 < a_2 < \ldots < a_h$. Now perform a double induction on (n, h). If $a_h = n + h$, then π is a bijection: $\{1, \ldots, n\} \to \{1, \ldots, n + h - 1\} - \{a_1, \ldots, a_{h-1}\}$ so we reduce to the case for $(n, h - 1)$. If $a_h < n + h$, then for $i < n$ we have $\pi(i) < n + h$. Since $\{1, \ldots, n + h - 1\}$ contains $n - 1$ elements of A , we must have $\pi(n) = n + h$, and π gives a bijection: $\{1, \ldots, n - 1\} \to \{1, \ldots, n - 1 + h\} - \{a_1, \ldots, a_h\}$, so we reduce to the case $(n - 1, h)$.