Problemen

| Problem Section

Problem 38

Let $S = 4, 8, 9, 16, 25, 27, 32, 36, ...$ be the set of positive integers that are proper powers. Evaluate the sum $\sum_{s\in S} \frac{1}{s-1}$.

Problem 39 (M. Bencze)

Let $f\colon [0,1] \to \mathbf{R}$ be a differentiable function for which $f(x^2) + f(y^2) \le 2f(\sqrt{xy})$. Prove that $f(1) - f(\frac{1}{e}) \le \int_0^1 \sqrt{x} f'(x) dx$.

Problem 40 (Matthé van der Lee)

For $t, m \in \mathbf{N}$, let $b_m(t) = \binom{tm-1}{t-1}$, let $a(t)$ be a sequence of natural numbers and let $n \in \mathbf{N}$ be fixed. Verify the equivalence of the following statements:

- for all $t|n$ we have that $\sum_{d|t} a(d) = 0 \mod t$,
- for all $t|n$ there is an $m \in \mathbf{N}$ such that $\sum_{d|t} a(d) b_m(t/d) = 0 \mod t$,
- for all $t|n$ and $m \in \mathbf{N}$ we have that $\sum_{d|t} a(d) b_m(t/d) = 0 \mod t$.

The previous issue of Nieuw Archief presented a solution to Problem 29, but Lute Kamstra has pointed out an error. So his problem remains open.

Problem 29 (Lute Kamstra)

Let $n \in \mathbf{N}$, $h \in \mathbf{N}_0$ and let A be a subset of $\{1, 2, \dots, n+h\}$ of size n. Count the number of bijective maps $\pi : \{1, 2, \dots, n\} \to A$ such that $i \le \pi(i) \le i + h$ for all $1 \le i \le n$.

Solutions to volume 3, number 2 (March/June 2002)

Problem 30 (H.J.A. Duparc †)

In general, if you interchange the hands of a clock you don't get a valid time. How many times a day can you interchange the hands of a clock and still get a valid time?

Solution by F.J.M. Barning, Marc Bremer, Ruud Jeurissen, Ton Kool, Karel Post, Ed Schaefer. Karel Post remarks that this is an old problem. If the position of the big hand is at an angle θ, then the position of the small hand is at 12θ. You can change the hands if and only if $144\theta = \theta \mod (2\pi)$ or $143\theta = 0 \mod (2\pi)$, which has 143 solutions. Since a day has 24 hours, the answer is 286 times a day. One could also consider clocks with three hands.

Problem 31 (Sander Dahmen)

Define for each positive integer n the regular $(n+1)$-dimensional solid $R_n = \{x \in \mathbf{R}^{n+1}\colon \sum_{i=1}^{n+1} |x_i| \le 2\}$ and the n-dimensional hyperplane $V_n = \{x \in \mathbf{R}^{n+1}\colon \sum_{i=1}^{n+1} x_i = 0\}$. Compute the n-dimensional volume of $R_n \cap V_n$.

Solution Sander Dahmen shows that $\mathrm{Vol}_n(R_n \cap V_n) = \sqrt{n+1}\frac{(2n)!}{n!^3}$. Let $P_n \subset \mathbf{R}^n$ be the projection of $R_n \cap V_n$ onto the first n coordinates. It is easily checked that the volume of $R_n \cap V_n$ is equal to $\sqrt{n+1}$ times the volume of P_n. Now let $x = (x_1, \dots, x_n) \in \mathbf{R}^n$. A simple calculation shows that $x \in P_n$ iff

$$\sum_{\substack{i=1 \\ x_i>0}}^{n} x_i \le 1 \text{ and } |\sum_{\substack{i=1 \\ x_i<0}}^{n} x_i| \le 1. \qquad (1)$$

Let $I \subset \{1, 2, \dots, n\}$ and define $K_I = \{(x_1, \dots, x_n) \in \mathbf{R}^n\colon x_i \ge 0 \text{ for } i \in I \text{ and } x_i \le 0 \text{ for } i \notin I\}$. Adding up the volumes of P_n contained in K_I for all 2^n possible sets I gives

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

Editor:
R.J. Fokkink
Technische Universiteit Delft
Faculteit Wiskunde
P.O. Box 5031
2600 GA Delft
The Netherlands
r.j.fokkink@its.tudelft.nl

the volume of P_n. By symmetry the volume of P_n contained in K_I only depends on the number of elements in I. For p elements, there are $\binom{n}{p}$ possible sets I. We compute $\mathrm{Vol}_n(P_n \cap K_I)$ for $I = \{1, 2, \ldots, p\}$. Define the 'm-dimensional hyperpyramid'

$$Y_m = \{(x_1, \ldots, x_m) \in \mathbf{R}^m : x_1, \ldots, x_m \geq 0 \text{ and } \sum_{i=1}^{m} x_i \leq 1\},$$

which has volume $\frac{1}{m!}$. Identify $\mathbf{R}^n$ with $\mathbf{R}^p \times \mathbf{R}^{n-p}$. Then (1) implies that $K_I \cap P_n = Y_p \times (-Y_{n-p})$. So $\mathrm{Vol}_n(K_I \cap P_n) = \mathrm{Vol}_p(Y_p) \cdot \mathrm{Vol}_{n-p}(Y_{n-p}) = \frac{1}{p!(n-p)!}$ Summing over all possible sets I gives:

$$\mathrm{Vol}_n(P_n) = \sum_{p=0}^{n} \binom{n}{p} \frac{1}{p!(n-p)!} = \frac{1}{n!} \sum_{p=0}^{n} \binom{n}{p}^2. \tag{2}$$

Using the identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ and multiplying by $\sqrt{n+1}$ we conclude from (2):

$$\mathrm{Vol}_n(R_n \cap V_n) = \sqrt{n+1}\frac{1}{n!}\binom{2n}{n} = \sqrt{n+1}\frac{(2n)!}{n!^3}.$$

Problem 32
Show that you can cover a square by three smaller squares, but not by two smaller squares.

Solution by Ruud Jeurissen, Kee Wai Lau and Lin Wensong. You can cover a square, using two smaller squares to cover two adjacent edges. The remainder can be covered by a smaller square. Now we prove that two squares are not enough. Let $ABCD$ be a unit square and suppose that is covered by two smaller squares V and W. Enlarging the smallest one we suppose that both have sides $1 - \epsilon$ for some $\epsilon > 0$. Since V nor W can cover both A and C, or both B and D, each must cover two adjacent vertices. Say V covers A and B. Then W covers C and D. There is a $\delta > 0$ such that W does not cover points at distance $< \delta$ from A or from B. So V covers a part AP of AD and a part BQ of BC. Let $p = |AP|$ and $q = |BQ|$. Since V and W cover the square $p + q \geq 1$. This implies that a square $EFGH$ with sides $1 - \epsilon$ can contain a trapezoid $KLMN$ with $|KL| = 1$ and ML of lengths p and NK of length q perpendicular to KL. By a translation of the trapezoid and a relabelling of $EFGH$ we may suppose that K is on EF and L is on FG. But then the (possibly) larger trapezoid fits that we get by supposing that M is on GH and that N is on EH. Then let $\angle ENK = \angle FKL = \angle GLM = \alpha$. We have

$$q \sin\alpha = |EK| = 1 - \epsilon - |FK| = 1 - \epsilon - \cos\alpha,$$
$$p \cos\alpha = |GL| = 1 - \epsilon - |FL| = 1 - \epsilon - \sin\alpha.$$

Multiplying the first equality by $\cos\alpha$ and the second one by $\sin\alpha$ and adding we get

$$(p+q)\sin\alpha\cos\alpha = (1-\epsilon)(\cos\alpha + \sin\alpha) - 1.$$

But since $p + q \geq 1$ and $\sin\alpha\cos\alpha + 1 \geq \cos\alpha + \sin\alpha$ (square both sides) we have a contradiction.

Problem 33
Show that you can cover an n-dimensional cube by $n + 1$ smaller cubes.

Solution By Lin Wensong. His proof is quite intricate and will be published on Mathematics Web (http://www.mathematicsweb.org).

Problem 34
Can you cover an n-dimensional cube by n smaller cubes?

Solution This problem remains open.