Problemen

| Problem Section

Edition 2020-3 We received solutions from Brian Gilding, Pieter de Groen, Marco Pouw and Ludo Pulles.

Problem 2020-3/A (proposed by Onno Berrevoets)

Let $f:(-1,1)\to\mathbb{R}$ be a function of class C^∞, i.e., all higher derivatives of f exist on $(-1,1)$. Let $c\ge 0$ be a real number. Suppose that for all $x\in(-1,1)$ and all $n\in\mathbb{Z}_{\ge 0}$ we have $f^{(n)}(x)\ge -c$. Also assume that for all $x\in(-1,0]$ we have $f(x)=0$. Prove that f is the zero function.

Solution We received solutions from Brian Gilding, Pieter de Groen and Marco Pouw. This solution is based on the one by Brian Gilding, who not only gives a very concise solution, but also shows that some of the assumptions can be weakened.

Since $f\in C^\infty(-1,1)$ and $f\equiv 0$ in $(-1,0]$, $f^{(n)}(0)=0$ for every $n\in\mathbb{Z}_{\ge 0}$. Consequently, for arbitrary $x\in(0,1)$ and $n\in\mathbb{Z}_{\ge 2}$, Taylor's Theorem (or repeated integration by parts, following the proof by Pieter de Groen) gives

$$f(x)=\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t)dt\ge -c\int_0^x\frac{(x-t)^{n-1}}{(n-1)!}dt=-c\frac{x^n}{n!}.$$

Likewise,

$$f'(x)=\int_0^x\frac{(x-t)^{n-2}}{(n-2)!}f^{(n)}(t)dt,$$

and this gives us

$$f(x)-\frac{x}{n-1}f'(x)=-\int_0^x\frac{t(x-t)^{n-2}}{(n-1)!}f^{(n)}(t)dt\ge c\int_0^x\frac{t(x-t)^{n-2}}{(n-1)!}dt=\frac{c}{n-1}\frac{x^n}{n!}.$$

Passage to the limit $n\to\infty$ yields $f(x)=0$ for all such x.

The assumption $f^{(n)}\ge -c$ in $(-1,1)$ for every $n\in\mathbb{Z}_{\ge 0}$ for some nonnegative real number c can be relaxed to $\pm f^{(n)}\le n!g_n$ for every $n\in\mathbb{Z}_{\ge 0}$ for a sequence of nonnegative functions $\{g_n\colon n\in\mathbb{Z}_{\ge 0}\}\subset L^\infty_{\mathrm{loc}}(-1,1)$ with the property $x^n\|g_n\|_{L^\infty(-x,x)}\to 0$ as $n\to\infty$ for all $x\in(0,1)$. Furthermore, given that $f^{(n)}(0)=0$ for every $n\in\mathbb{Z}_{\ge 0}$, it is not necessary to suppose that $f\equiv 0$ in $(-1,0)$. This can be shown analogously to $f\equiv 0$ in $(0,1)$.

Problem 2020-3B (proposed by Onno Berrevoets)

Consider the map $f:\mathbb{Z}^2_{\ge 0}\to\mathbb{Z}^2_{\ge 0},\ \ (a,b)\mapsto(2\min\{a,b\},\max\{a,b\}-\min\{a,b\})$.
We call $(a,b)\in\mathbb{Z}^2_{\ge 0}$ *equipotent* if there exists $n\in\mathbb{Z}_{\ge 0}$ such that $f^n(a,b)=(x,x)$ for some $x\in\mathbb{Z}_{\ge 0}$ (where $f^n=f\circ\cdots\circ f$). Show that $(a,b)\in\mathbb{Z}^2_{\ge 1}$ is equipotent if and only if $\frac{a+b}{\gcd(a,b)}$ is a power of 2.

Solution We received solutions by Pieter de Groen and Ludo Pulles. This solution is based on the one by Pieter.

It is clear that $f(ca,cb)=cf(a,b)$ for all non-negative integers a, b, k, and it follows that (ca,cb) is equipotent if and only if (a,b) is. So it suffices to show that for $a,b\ge 1$ relatively prime, we have (a,b) is equipotent if and only if $a+b=2^k$ for some $k\ge 1$.

$\Leftarrow$: Suppose that $a,b\in\mathbb{Z}_{\ge 1}$ are relatively prime and satisfy $a+b=2^k$. If $k=1$, we have $(a,b)=(1,1)=f^0(1,1)$ is equipotent. If $k>1$ and $(c,d):=f(a,b)$, then $c=2\min(a,b)$ is even, and hence so is d because $c+d=a+b$ is even. Hence (a,b) is equipotent with sum 2^k if and only if $(\frac{c}{2},\frac{d}{2})$ is equipotent with sum 2^{k-1}. Note that $\frac{c}{2}$ and $\frac{d}{2}$ are relatively prime because $\gcd(2\min\{a,b\},\max\{a,b\}-\min\{a,b\})$ can only take on the values $\gcd(a,b)$ or $2\gcd(a,b)$. We can conclude (a,b) is equipotent by induction.

$\Rightarrow$: Conversely, suppose that $a,b\in\mathbb{Z}_{\ge}1$ are relatively prime with $a+b$ not a power of 2. Note that the sum of (a,b) is invariant under f, because if $f(a,b)=(p,q)$, we have $p+q=\max(a,b)+\min(a,b)=a+b$. If $a+b$ is odd, then the same is true for $f^n(a,b)$, so (a,b) is not equipotent. Suppose $a+b$ is even. Because a,b are relatively prime, both a

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and b are odd. Now similar to the above, if $f(a,b)=(c,d)$, then both c and d are even, and (a,b) is equipotent if and only if $(\frac{c}{2},\frac{d}{2})$ is. Since $\frac{c}{2}+\frac{d}{2}=\frac{a+b}{2}$ and $\frac{c}{2},\frac{d}{2}$ are relatively prime, repeating this procedure eventually results in a pair with odd element-sum, which is not equipotent. Hence (a,b) was not equipotent either.

Problem 2020-3/C* (folklore)

Uncle Donald cuts a 3 kg piece of cheese in an arbitrary, finite number of pieces of arbitrary weights. He distributes them uniformly randomly among his nephews Huey, Dewey and Louie. Prove or disprove: the probability that two of the nephews each get strictly more than 1 kg is at most two thirds.

Solution This problem remains open. This is a Star Problem for which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of € 100.