Problemen

| Problem Section

Edition 2019-1 We received solutions from Paul Hutschemakers, Hendrik Reuvers and Hans Samuels Brusse.

Problem 2019-1/A (folklore)

Three gamblers each select a non-negative probability distribution with mean 1. Say these distributions are F, G, H. Then x is sampled from F, y is sampled from G, and z is sampled from H. Biggest number wins. What distributions should the gamblers choose?

Solution Suppose two of the gamblers choose the same distribution function: $\Phi(t)=\sqrt{t/3}$ on the interval $[0,3]$. What should the other gambler do? If she flips a coin and says 3 for H and 0 for T, then she needs a coin that has probability $\frac{1}{3}$ for H in order to comply with the rules of the game. She wins one third of the time. She could also try a fair coin and say 2 for H and 0 for T. What is the probability that she wins with this strategy? She needs to beat the maximum of two numbers that are sampled from the $\sqrt{t/3}$-distribution. The distribution of the maximum is $M(t)=t/3$. Therefore, the probability that 2 is the winning number is $\frac{2}{3}$. The probability that H comes up is $\frac{1}{2}$. Again, she wins one third of the time. Of course, the other gambler may try other coins and other numbers. We leave it to the reader to verify that each of them have a probability of $\frac{1}{3}$ of winning. The other gambler should not use a number bigger than three. If she is overly concerned and says 4 just to be on the safe side, she will not win one third of the time.
Any probability distribution on $[0,3]$ with mean one is a mixture of coins with mean one. Therefore, the other gambler may just as well sample from $\Phi(t)$ to win one third of the time. If all three gamblers sample from this distribution, each of them wins a third of the time and has no reason to deviate. We solved the game. As always, it is a bit of a mystery how we found this solution. There is no good algorithm to find a Nash equilibrium.
This game is taken from a recent paper by Steve Alpern and John Howard, 'Winner-take-all-games', *Operations Research* 65, 2017. They solve the n-player version and show that the solution is unique. Apparently, it remains an open problem to solve the game if different players have different means. Suppose we have a new Da Vinci coming up at Christie's and three different Saudi royals with three different means want to buy it. In a one-shot auction, how should they bid?

Problem 2019-1/B (proposed by Hendrik Lenstra)

For given $m\in\mathbb{Z}_{\geq 3}$, consider the regular m-gon inscribed in the unit circle. We denote the surface of this m-gon by A_m. Suppose m is odd. Prove that $2A_m$ and A_{2m} have the same minimal polynomial.

Solution We find $A_m=\frac{m}{2}\sin(\frac{2\pi}{m})$ by basic geometry, so $2A_m=m\cdot\sin(\frac{2\pi}{m})$ and $A_{2m}=m\cdot\sin(\frac{2\pi}{2m})$. Let ζ be a primitive $2m$-th root of unity. If we embed in $\mathbb{C}$ by taking $\zeta=\cos(\frac{2\pi}{2m})+i\sin(\frac{2\pi}{2m})$ in $\mathbb{C}$, we find $A_{2m}=\frac{m}{2i}(\zeta+\zeta^{-1})$ and $2A_m=\frac{m}{2i}(\zeta^2+\zeta^{-2})$. Since m is odd, ζi is a primitive $4m$-th root of unity, and so is $\zeta^2 i$. We consider the field $\mathbb{Q}(\zeta,i)=\mathbb{Q}(\zeta i)$. Observe that the field automorphism defined by sending ζi to $\zeta^2 i$ sends ζ to $-\zeta^2$ and i to $-i$ (this can be verified using $i=\pm(\zeta i)^m$ and $\zeta=\pm(\zeta i)^{m+1}$). Therefore this automorphism sends A_{2m} to $2A_m$ (implicitly using the earlier embedding into $\mathbb{C}$). Since automorphisms preserve minimal polynomials, it follows that A_{2m} and $2A_m$ have the same minimal polynomial.

Problem 2019-1/C (proposed by Nicky Hekster)

Let n be a prime number. Show that there are no groups with exactly n elements of order n. What happens with this statement if n is *not* a prime number?

Solution Solutions were submitted by Hans Samuels Brusse, Hendrik Reuvers and Paul Hutschemakers. The solution below is based on the solution by Hans.

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Oplossingen | Solutions

Suppose G is a group with exactly n elements of prime order n. Let g be a group element of G of order n. Then $H_1 = \{1, g, g^2, \ldots, g^{n-1}\}$ is the subgroup generated by g and all $n-1$ elements $g, g^2, \ldots, g^{n-1}$ have order n. Since n is prime any of these elements can serve as generator for H_1.

Since G contains n different elements of order n by assumption, there must be exactly one more. Assume h is this last element, then h is not in H_1 and it will generate a different subgroup $H_2 = \{1, h, h^2, \ldots, h^{n-1}\}$. Note that $h, h^2, \ldots, h^{n-1}$ are distinct and do not belong to H_1, since this would imply $h \in H_1$. This gives us $2(n-1) \geq n$ distinct elements of order n, which leads to a contradiction unless $n = 2$.

In the case $n = 2$, we have distinct elements g, h of order 2. Note that ghg^{-1} has order 2 as well. Clearly, it cannot equal g, so it must equal h. However, this means g and h commute, and we find that the element gh is of order two and not equal to either g or h. So we find a contradiction in this case as well.

If n is not prime, the statement is false. For example, the abelian group $C_4 \times C_2$ (with C_k the cyclic group of order k) contains four elements of order four.

In the paper 'Finite groups that have exactly n elements of order n' by Carrie E. Finch, Richard M. Foote, Lenny Jones and Donald Spickler, Jr., *Mathematics Magazine* 75(3) (June 2002), pp. 215–219, the *finite* groups with the mentioned property are classified.