Problem

| Problem Section

Edition 2018-2 We received solutions from Pieter de Groen (Brussel), Alexander van Hoorn (Loenersloot), Thijmen Krebs (Nootdorp), Tejaswi Navilarekallu (Bengaluru), Sander Rieken (Arnhem), Hendrik Reuvers (Maastricht), Yan Zhao (Leiden). The book tokens go to Thijmen Krebs, Alexander van Hoorn and Yan Zhao.

Problem 2018-2/A

Let $n>2$ be an odd integer and let C be an embedding of the circle in $\mathbb{R}^n$. That is, $C=f([0,1])$, where $f:[0,1]\to\mathbb{R}^n$ is continuous, $f(0)=f(1)$, and f is injective on $[0,1)$. Show that there is an affine hyperplane in $\mathbb{R}^n$ that contains at least $n+1$ points from C.

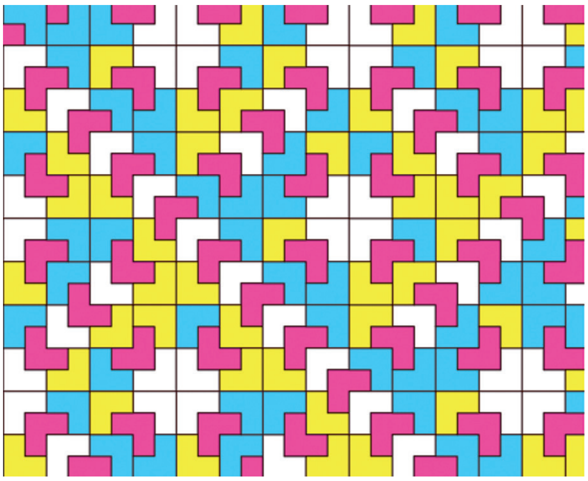
Solution Solved by Thijmen Krebs, Tejaswi Navilarekallu, Sander Rieken, Hendrik Reuvers, Yan Zhao. The idea is that C crosses a hyperplane in an even number of points and that any n points on C are in a hyperplane. Since n is odd, one expects one more crossing. The problem is that not all n points need to be crossings. There can also be turning points. To deal with this, the hyperplane needs to be perturbed. Thijmen Krebs deals with this most succinctly and we follow his solution.

Pick an affine subspace H spanned by any set S of n distinct points from C. If H contains another point from C, or H is not an affine hyperplane, we are done. Since f crosses H an even number of times, it must touch H without crossing in some $f(x)\in S$. That is, there are $0<a<x<b<1$ such that $f([a,b])$ lies entirely on one side of H and intersects H only once. By replacing $f(x)$ in the spanning set S with some sufficiently close $f(y)f([a,x])$, we replace S by a set S', deleting a turning point. Getting rid of all the turning points, we eventually find a hyperplane such that the n elements of S are crossings.

Problem 2018-2/B

Place coins on the vertices of the lattice $\mathbb{Z}^2$, all showing heads. You are allowed to flip coins in sets of three at positions $(m,n),(m,n+1)$ and $(m+1,n)$ where m and n can be chosen arbitrarily. Is it possible to achieve a position where two coins are showing tails and all others show heads using finitely many moves?

Solution Solved by Thijmen Krebs, Alexander van Hoorn, Tejaswi Navilarekallu, Hendrik Reuvers, Yan Zhao. The answer is no. Here is Tejaswi Navilarekallu's solution. Suppose we make finitely many moves. Let the choice of the pair (m,n) for each of these moves be $(m_1,n_1),(m_2,n_2),\dots,(m_k,n_k)$. Without loss of generality, we may assume that $(m_i,n_i)\neq(m_j,n_j)$ for all $1\le i<j\le k$. Further, we may assume that $m_1\le m_2\le\cdots\le m_k$ and if $m_i=m_{i+1}$ for some $1\le i\le k-1$ then $n_i<n_{i+1}$. The coin at (m_1,n_1) is turned exactly once, so is the coin at (m_k+1,n_k). Let $1\le i\le k$ be such that $n_i=\max_{1\le j\le k}n_j$. Then the coin at (m_i,n_i+1) is also flipped at most once. Clearly, these three coins are distinct, so there are at least three coins that are showing tails.



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Oplossingen | Solutions

Thijmen Krebs says that the positions $(m,n),(m,n+1)$ and $(m+1,n)$ form a *chair*, in analogy with the well-known chair tiling of the plane. In the tiling, chairs can be turned. If we could also do that in our problem, it would have been easy to come up with two coins showing tails in just two moves. Is it possible to come up with a sequence of moves such that only one coin shows tails?
Alexander van Hoorn observes that it is more convenient to describe problem B by polynomials. Let $I \subset \mathbb{F}_2[X,Y]$ be the ideal generated by $1+X+Y$. Does I contain a polynomial of length two, i.e., of the form $X^aY^b+X^cY^d$? The answer is no since modulo I this is $X^a(1+X)^b+X^c(1+X)^d=0$, which implies $a=c$ and $b=d$.
What if we are allowed to turn the chair upside down? Let J be the ideal generated by $1+X+Y$ and $X+Y+XY$. Does it contain a monomial? The general problem of finding a polynomial of shortest length within an ideal is hard. Harm Derksen and David Masser solved it and their result generalizes the Skolem–Mahler–Lech theorem on linear recurrence. They have written a sequence of papers entitled *Linear equations over multiplicative groups, recurrences, and mixing*, one of which appeared in *Indagationes Mathematicae* in 2015.

Problem 2018-2/C

Say that a natural number is k-repetitive if its decimal expansion is a concatenation of k equal blocks. For instance, 1010 is 2-repetitive and 666 is 3-repetitive. Let R_k be the set of all k-repetitive numbers. Determine its greatest common divisor.

Solution Solved by Pieter de Groen, Thijmen Krebs, Alexander van Hoorn, Tejaswi Navilarekallu, Hendrik Reuvers, Yan Zhao. Problem C is taken from a recent paper by Daniel Kane, Carlo Sanna and Jeffrey Shallit, 'Waring's problem for binary powers', arXiv:1801.04483. Waring's problem is to determine the minimal number $g(k)$ such that every natural number is the sum of $g(k)$ k-th powers. Kane, Sanna and Shallit consider the analogous problem for sums of k-repetitive numbers.
The greatest common divisor is

$$\gcd\left(\frac{10^k-1}{9},k\right)$$

We more or less follow Yan Zhao's solution. Let r_k be the gcd of the set R_k. Consider the set

$$Y_k=\left\{y_a=1+10^a+\cdots+10^{(k-1)a}=\frac{10^{ka}-1}{10^a}: a\in\mathbb{Z}_{>0}\right\}.$$

All elements of R_k are multiples of elements of Y_k, so $\gcd(Y_k)\mid r_k$. Conversely, if a repetitive number has blocks of length a, then it is a multiple of y_a. Clearly, a-digit numbers have gcd equal to one, so $r_k=\gcd(Y_k)$.
Observe that

$$y_a=1+10^{a \bmod k}+\cdots+10^{(k-1)a \bmod k} \pmod{10^k-1}$$

and that y_1 divides 10^k-1. In particular $y_k=k \pmod{y_1}$ which implies that r_k divides $\gcd(y_1,k)$. To finish the proof we need to show that $\min\{\mathrm{ord}_p(y_1),\mathrm{ord}_p(k)\}\le \mathrm{ord}_p(y_a)$ for all primes p. If p does not divide 10^a-1, then $\mathrm{ord}_p(y_a)=\mathrm{ord}_p(10^{ka}-1)\ge \mathrm{ord}_p(y_1)$ since $10^{ka}-1$ is a multiple of y_1. If p does divide 10^a-1, let $10^a=1 \pmod{p^d}$. By the binomial theorem $10^{ap}=1 \pmod{p^{d+1}}$ and by induction $10^{ap^j}=1 \pmod{p^{d+j}}$. It follows that $\mathrm{ord}_p(y_a)=\mathrm{ord}_p(10^{ka}-1)-\mathrm{ord}_p(10^a-1)\ge \mathrm{ord}_p(k)$. Which finishes the proof. Some of the other solvers point out that the final step in this proof is called the 'lifting-the-exponent lemma'.