Problemen

| Problem Section

Edition 2016-2 We received solutions from Hendrik Reuvers, Pieter de Groen en Toshihiro Shimizu.

Problem 2016-2/A (folklore)

Denote for all positive rational numbers x by $f(x)$ the minimum number of 1's needed in a formula for x involving only ones, addition, subtraction, multiplication, division and parentheses. For example, $f(1)=1$, and $f(\frac{1}{3})=4$, as $\frac{1}{3}=\frac{1}{1+1+1}$ and as no such formula exists with at most three 1's. Note that $f(11)\neq 2$ (concatenation of ones is not allowed).
Moreover, denote for all positive rational numbers x by $h_2(x)$ the number $\log_2(p)+\log_2(q)$, where $\log_2$ denotes the base-2 logarithm, and where p, q are positive integers such that $x=\frac{p}{q}$ and $\gcd(p,q)=1$.
Show that for all x, we have

$$f(x)>\tfrac{1}{2}h_2(x).$$

Solution We received a solution from Toshihiro Shimizu, to whom the book token is awarded, and the solution of whom the following is based on.
Let x be a positive rational number, and let p, q be positive integers such that $x=\frac{p}{q}$ and $\gcd(p,q)=1$. We show by induction on $f(x)$ that $p,q\leq 2^{f(x)-1}$; from this it will follow that $2f(x)>2f(x)-2\geq\log_2 p+\log_2 q=h_2(x)$, as desired.
First note that $x=1$ is the only rational number with $f(x)=1$, and in this case $p=q=1=2^{f(x)-1}$. So suppose that n is a positive integer and that for any x with $f(x)\leq n$ written as $\frac{p}{q}$ with p, q positive integers with $\gcd(p,q)=1$, we have $p,q\leq 2^{f(x)-1}$. Suppose that $f(x)=n+1$. Then x can be written in one of the forms x_1+x_2, x_1-x_2, $x_1\cdot x_2$, $\frac{x_1}{x_2}$with x_1, x_2 positive rational numbers with $f(x_1)+f(x_2)=f(x)=n+1$ (so that $f(x_1), f(x_2)\leq n$).
Write $x_i=\frac{p_i}{q_i}$where p_i and q_i are positive integers such that $\gcd(p_i,q_i)=1$, for $i=1,2$.
If $x=x_1+x_2$ or $x=x_1-x_2$, then we find that $x=\frac{p_1q_2\pm p_2q_1}{q_1q_2}$, so $p,q\leq 2^{1+f(x_1)-1+f(x_2)-1}=2^n$ $=2^{f(x)-1}$. If $x=x_1\cdot x_2$, then we find that $x=\frac{p_1p_2}{q_1q_2}$, so $p,q\leq 2^{f(x_1)-1+f(x_2)-1}=2^{n-1}<2^{f(x)-1}$.
The same argument shows that if $x=\frac{x_1}{x_2}$, that then $p,q<2^{f(x)-1}$. Hence we always have $p,q\leq 2^{f(x)-1}$, and we are done.

Problem 2016-2/B (folklore)

Suppose that there are $N\geq 2$ players, labeled $1,2,\ldots,N$, and that each of them holds precisely $m\geq 1$ coins of value 1, m coins of (integer) value $n\geq 2$, m coins of value n^2, et cetera. A *transaction* from player i to player j consists of player i giving a finite number of his coins to player j. We say that an N-tuple $(a_1,a_2,\ldots,a_N)$ of integers is (m,n)-*payable* if $\sum_{i=1}^N a_i=0$ and after a finite number of transactions, the i-th player has received (in value) a_i more than he has given away.
Show that for every N-tuple $(a_1,a_2,\ldots,a_N)$ with $\sum_{i=1}^N a_i=0$ to be (m,n)-payable, it is necessary and sufficient that $m>n-\frac{n}{N}-1$.

Solution We received solutions from Pieter de Groen and Toshihiro Shimizu. The book token goes to Pieter de Groen. Both solutions shared the same idea, the following solution is based on that of Toshihiro Shimizu.
First observe that if an N-tuple $(a_1,a_2,\ldots,a_n)$ is (m,n)-payable, the number of coins of value 1 that player i receives is a_i modulo n, since coins of higher value are of value divisible by n.
Also observe that $m>n-\frac{n}{N}-1$ is equivalent to $mN\geq(n-1)(N-1)$.
Note that if $m\leq n-\frac{n}{N}-1$ (so $m<n$), or equivalently, $mN<(n-1)(N-1)$, then the tuple

$$(n-m-1,n-m-1,\ldots,n-m-1,-(N-1)(n-m-1))$$

is not (m,n)-payable; every player up to player $N-1$ has to either receive at least $n-m-1$ coins of value 1, or give away at least $m+1$ coins of value 1. The latter is clearly impossible. This means that the last player must give away at least $(N-1)(n-m-1)$ coins of value 1. However, $(N-1)(n-m-1)=(n-1)(N-1)-m(N-1)>m$, so this is also impossible. Therefore the condition $m>n-\frac{n}{N}-1$ is necessary.
Next, we show that the condition $m>n-\frac{n}{N}-1$ is sufficient. Let $(a_1,\ldots,a_N)$ be any N-tuple

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of integers with $\sum_i a_i = 0$. Assume that $m > n - \frac{n}{N} - 1$. We show that $(a_1,\dots,a_N)$ is (m,n)-payable, by induction on $\max_i |a_i|$.
If $\max_i |a_i| \le 1$, then all a_i lie in $\{-1,0,1\}$, and as the sum of the a_i's is zero, the tuple is (m,n)-payable (by having every player with negative a_i pay one coin of value 1, and every player with positive a_i receive one coin of value 1). So assume that $A > 1$, and that all tuples $(a_1,\dots,a_N)$ with zero sum and $\max_i |a_i| < A$ are (m,n)-payable.
Let $(a_1,\dots,a_N)$ be a tuple with $\max_i |a_i| = A$. Let r_i denote the remainder of a_i on division by n. We assume without loss of generality that $n > r_1 \ge r_2 \ge \cdots \ge r_N \ge 0$. Note that $\sum_i r_i$ is a multiple of n, say $\sum_i r_i = kn$. We show that $n - r_1,\dots,n - r_k \le m$, so that player i can pay $n - r_i$ coins of value 1 for $i = 1,2,\dots,k$.
Note that

$$kn = \sum_i r_i = (r_1 + \cdots + r_{k-1}) + (r_k + \cdots + r_N) \le (N-k+1)\,r_k + (k-1)\,n.$$

Therefore $r_k \ge \frac{n}{N-k+1} \ge \frac{n}{N}$, so $n - r_k = n - \frac{n}{N} < m + 1$, from which we deduce that $n - r_k \le m$, and therefore also that $n - r_i \le m$ for all $i = 1,2,\dots,k$. So by having, for $i = 1,2,\dots,k$, player i pay $n - r_i$, and for all other i, player i receive r_i (which is possible by the above and since $\sum_i r_i = kn$), we see that the tuple $(a_1,\dots,a_N)$ is (m,n)-payable if $(a_1 + n - r_1,\dots,a_k + n - r_k, a_{k+1} - r_{k+1},\dots,a_N - r_N)$ is (m,n)-payable using only coins of value n or higher, and therefore if the tuple

$$(a'_1,\dots,a'_N) = \left(\tfrac{1}{n}(a_1 + n - r_1),\dots,\tfrac{1}{n}(a_k + n - r_k),\tfrac{1}{n}(a_{k+1} - r_{k+1}),\dots,\tfrac{1}{n}(a_N - r_N)\right)$$

is (m,n)-payable.
We finish the induction by showing that $\max_i |a'_i| < A$, so that by the induction hypothesis, the tuple $(a'_1,\dots,a'_N)$is indeed (m,n)-payable. Note that $a'_i = \left\lceil \frac{a_i}{n} \right\rceil$ or $a'_i = \left\lfloor \frac{a_i}{n} \right\rfloor$, so $|a'_i| \le \left\lceil \frac{|a_i|}{n} \right\rceil$. Since $\left\lceil \frac{|a_i|}{n} \right\rceil \le |a_i|$ with equality if and only if $|a_i| = 0$ or $|a_i| = 1$, it follows that $|a'_i| < |a_i|$ for all i with $|a_i| \ge 2$. Since $A \ge 2$, it follows that $\max_i |a'_i| < A$, and we are done.

Problem 2016-2/C (proposed by Wouter Zomervrucht)

For each integer $n \ge 1$ let c_n be the largest real number such that for any finite set of vectors $X \subset \mathbb{R}^n$ with $\sum_{v\in X} |v| \ge 1$ there exists a subset $Y \subseteq X$ with $|\sum_{v \in Y} v| \ge c_n$. Prove the recurrence relation

$$c_1 = \tfrac{1}{2}, \qquad c_{n+1} = \tfrac{1}{2\pi n c_n}.$$

Solution We received solutions from Hendrik Reuvers and Toshihiro Shimizu. The book token is awarded to Hendrik Reuvers. Both solutions shared the same idea, the following solution is based on the one sent in by the proposer.
For $n \ge 0$, let $D^n \subset \mathbb{R}^n$ be the closed unit ball and $S^n \subset \mathbb{R}^{n+1}$ the unit sphere. Denote by v_n the volume of D^n and by s_n the surface area of S^n. We will show that $c_n = v_{n-1}/s_{n-1}$, then we are done by the relations $v_0 = 1$, $s_0 = 2$, $v_n = \frac{1}{n} s_{n-1}$, and $s_n = 2\pi v_{n-1}$.
First we make a computation. Let $n \ge 1$ and write $V_+ = \{x \in \mathbb{R}^n\colon x_n \ge 0\}$. For $r > 0$ we let rD^n be the closed radius r ball, then one has

$$\int_{rD^n \cap V_+} x_n \, dx = \int_0^r x_n (r^2 - x_n^2)^{\frac{n-1}{2}} v_{n-1} \, dx_n = \frac{v_{n-1}}{n+1} r^{n+1},$$

so

$$\int_{S^{n-1} \cap V_+} x_n \, dx = \frac{d}{dr}\left[\int_{rD^n \cap V_+} x_n \, dx\right]_{r=1} = v_{n-1}.$$

Now we turn to the problem. Take $n \ge 1$ and any collection $X \subset \mathbb{R}^n$ with $\sum_{v \in X} |v| \ge 1$. Let $Y \subseteq X$ be a subset for which the subsum $w = \sum_{v \in Y} v$ has maximal norm. Then Y must contain all $v \in X$ with $v \cdot w > 0$, and no $v \in X$ with $v \cdot w < 0$. In fact, $|w| = m(w/|w|)$, where for any $x \in S^{n-1}$we define

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$$m(x) = \sum_{v \in X \text{ with } v \cdot x \geq 0} v \cdot x.$$

It follows that $|w| = \max_{x \in S^{n-1}} m(x)$. For $v \in \mathbb{R}^n$ set $V_v = \{x \in \mathbb{R}^n : v \cdot x \geq 0\}$. (For instance, $V_+ = V_{(0,\ldots,0,1)}$.) We compute

$$\int_{S^{n-1}} m(x)\, dx = \sum_{v \in X} \int_{S^{n-1} \cap V_v} v \cdot x\, dx = \sum_{v \in X} |v| \int_{S^{n-1} \cap V_+} x_n\, dx \geq v_{n-1}.$$

Thus there is $x \in S^{n-1}$ where $m(x) \geq v_{n-1}/s_{n-1}$, hence $c_n \geq v_{n-1}/s_{n-1}$.
Conversely, let X consist of k vectors with lengths $1/k$ and directions distributed homogeneously over S^{n-1}. (There are several ways of doing this; thanks to Toshihiro Shimizu for pointing out [1].) As $k \to \infty$, the associated function m converges uniformly to a constant function; by the computations above, its constant value is v_{n-1}/s_{n-1}. So also $c_n \leq v_{n-1}/s_{n-1}$.

Reference

1 Eric W. Weisstein, Sphere Point Picking, http://mathworld.wolfram.com/SpherePointPicking.html.