Problemen

| Problem Section

Star problems. In the June 2008 edition of the NAW we revisited a selection of unsolved star problems. Whoever sent in a solution first before July 1, 2009 would receive a book token. In this and upcoming editions we will publish some of the solutions we have received.

Problem (Star) 2008-2/3 Let A and B be $n \times n$ matrices over $\mathbf{C}$. Suppose that $\lim_{k\to\infty}(A^k + B^k)$ exists. Show that there exists $M \in \mathbf{C}^{n\times n}$ such that $\lim_{k\to\infty} A^k - kM$ and $\lim_{k\to\infty} B^k + kM$ exist. Give necessary and sufficient conditions on A and B for M to be zero.

Solution This problem was solved by Alex Heinis and Wim Hesselink. As Wim Hesselink sent in a solution first, he will receive the prize. The following is based on both solutions. Clearly the matrix M is unique, if it exists. For any linear map $f\colon \mathbf{C}^n \to \mathbf{C}^n$, the vector space $\mathbf{C}^n$ is the direct sum of the generalized eigenspaces $E_{f,\lambda} = \ker(f-\lambda)^n$ for eigenvalues λ of f by the theory of Jordan normal forms.
Lemma 1. For any linear map f on $\mathbf{C}^n$ we have $\lim_k f^k = 0$ if and only if every eigenvalue λ of f satisfies $|\lambda| < 1$.
Proof. The only-if part being obvious, we assume that every eigenvalue λ of f satisfies $|\lambda| < 1$. Let λ be such an eigenvalue. Then the restriction f_λ of f to the generalized eigenspace $E_{f,\lambda}$ can be written as $\lambda \cdot \mathrm{id} + m$, with $m^n = 0$. We get $f_\lambda^k = \sum_{j<n} \binom{k}{j}\lambda^{k-j}m^j$, which tends to 0, because in each term $\binom{k}{j}$ only grows polynomially in k. We conclude that f tends to 0. □
Lemma 2. Every eigenvalue λ of A or B satisfies $|\lambda| < 1$ or $\lambda = 1$.
Proof. For two sequences $(X_k)_k$ and $(Y_k)_k$ of matrices we write $X_k \sim Y_k$ if $\lim_k(X_k - Y_k) = 0$. Set $C = \lim_k(A^k + B^k)$. Then we have

$$C - A^{k+1} \sim B^{k+1} \sim B(C - A^k) = BC - BA^k,$$

and therefore $\lim_k(B-A)A^k = (B-I)C$. Let $x \in \mathbf{C}^n$ be an eigenvector for A with eigenvalue λ. Then $\lambda^k(Bx - \lambda x) = (B-A)A^k x$ converges, namely to $(B-I)Cx$. We conclude that either $|\lambda| < 1$ or $\lambda = 1$, or $Bx = \lambda x = Ax$, in which case $2\lambda^k x = (A^k + B^k)x$ converges, and we also find $|\lambda| < 1$ or $\lambda = 1$. The statement for eigenvalues of B follows from symmetry. □
Let a and b denote the linear maps on $\mathbf{C}^n$ defined by multiplication by A and B respectively, whose eigenvalues are given in the previous lemma. Let $p, m, r\colon \mathbf{C}^n \to \mathbf{C}^n$ be the unique linear maps that equal 0, 0, and a, respectively, on the generalized eigenspaces $E_{a,\lambda}$ of a associated to eigenvalues λ with $|\lambda| < 1$, while their restrictions to $E_{a,1}$ equal id, $a - \mathrm{id}$, and 0 respectively. In other words, with respect to the decomposition

$$\mathbf{C}^n \cong E_{a,1} \oplus \left(\bigoplus_{|\lambda|<1} E_{a,\lambda} \right)$$

the maps p, m, r are given as

$$p = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}, \qquad m = \begin{pmatrix} a - \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

Then m is nilpotent, we have $a = p + m + r$, and the identities

$$p^2 = p, \quad pm = m = mp, \quad pr = mr = 0 = rm = rp, \quad \lim_k r^k = 0$$

hold, the latter by Lemma 1. Similarly, we may write $b = q + l + s$ where l is nilpotent and

$$q^2 = q, \quad ql = l = lq, \quad qs = ls = 0 = sl = sq, \quad \lim_k s^k = 0.$$

Of course the decomposition in generalised eigenspaces for a and b are not necessarily the same. We have

$$a^k + b^k = r^k + s^k + p + q + \sum_{1\le j<n} \binom{k}{j}(l^j + m^j).$$

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As this has a limit and $\lim_k r^k = \lim_k s^k = 0$, we find that all terms $l^j + m^j$ for $1 \leq j \leq n$ vanish. In particular, $l + m = 0$ and $l^2 + m^2 = 0$, so that $l = -m$ and $m^2 = 0$. We conclude $a^k = p + r^k + km$ and $b^k = q + s^k - km$, so that $\lim_k a^k - km = p$ and $\lim_k b^k + km = q$. The first statement of the problem follows for the matrix M associated to m. The matrix M is zero if and only if the restriction of a to the generalized eigenspace $E_{a,1}$ is the identity.

Problem (Star) 2008-2/11 Let V be the complex vector space of all functions $f : \mathbf{C} \to \mathbf{C}$. Let W be the smallest linear subspace of V with the properties:

- the function $f(z) = z$ belongs to W,
- for all $f \in W$, $|f| \in W$.

Does $f(z) = \overline{z}$ belong to W?

Solution The following solution is due to David Preiss (Warwick), and was communicated to us by Miklos Laczkovich. Since the solution was already known, there is no prize winner.

We will show that $f(z) = \overline{z}$ does not belong to W. We claim that it suffices to show that there is a complex vector space S of complex valued functions on the circle $\mathbf{R}/2\pi\mathbf{Z}$ with the properties that

- $h(x) = e^{ix}$ belongs to S,
- for all $h \in S$, $|h| \in S$,
- $h(x) = \cos(x)$ does not belong to S.

Indeed, if the function $f(z) = \overline{z}$ is in W, then the function

$$\frac{e^{ix} + f(e^{ix})}{2} = \cos(x)$$

belongs to S.

Construction of S. Let U be the family of regions $U = \{x + iy : \psi(x) < y < \phi(x)\}$, where $\phi, \psi : \mathbf{R} \to \mathbf{R}$ are continuous, $\psi \leq 0 < \phi$ and $\{x : \psi(x) = 0\}$ is locally finite in $\mathbf{R}$.

Let H be the set of functions F on $\mathbf{C}$ for which there is a region $U \in$ U so that F is holomorphic on U, and such that there is an $a < 1$ with $\limsup_{z \in U, |z| \to \infty} |F(z)|/|z|^a = 0$.

Let F be the set of continuous functions $f : \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{R}$ with the property that there exist a positive integer n and functions $F_1, \ldots, F_n \in$ H (with corresponding regions $U_1, \ldots, U_n \in$ U) and open intervals $I_1, \ldots, I_n$ covering the circle minus a finite number of points so that $\cos x \in U_j$ and $f(x) = F_j(\cos x)$ for all $1 \leq j \leq n$ and all $x \in I_j$.

Let S be the set of all functions $\mathbf{R}/2\pi\mathbf{Z} \to \mathbf{C}$ of the form $f + ig + ce^{ix}$ where $f, g \in$ F and $c \in \mathbf{C}$. The set S is a linear subspace of the complex vector space of all complex-valued functions on the circle.

Clearly $h(x) = e^{ix}$ is an element of S.

Proof that S *is closed under* $h \mapsto |h|$. Let h be a function in S, and write h as

$$h(x) = F_j(\cos x) + iG_j(\cos x) + ce^{ix}$$

on the open interval I_j, with $F_j, G_j \in$ H. Let $a_j < 1$ be such that

$$\limsup_{z \in U_j, |z| \to \infty} |F(z)|/|z|^{a_j} = 0 \text{ and } \limsup_{z \in U_j, |z| \to \infty} |G(z)|/|z|^{a_j} = 0.$$

Assuming, as we may, that $\sin x$ does not change sign on any I_j, we have that on each I_j, $|h(x)|^2 = H_j(\cos x)$ where H_j is a linear combination of 1, F_j^2, G_j^2, $F_j(z)z$, $F_j(z)\gamma(z)$, $G_j(z)z$, $G_j(z)\gamma(z)$, where γ is a suitable branch of $\sqrt{1 - z^2}$. Removing from I_j the finite set where $h(x) = 0$ we have that on each remaining interval $|h|$ coincides with a branch of $H_j^{1/2}$ and one verifies that $|h| \in$ F, where the constant can be taken to be $(a + 1)/2$.

Proof that $h(x) = \cos x$ *does not belong to* S. Assume that $\cos x = f(x) + ig(x) + ce^{ix}$ where $c \in \mathbf{C}$ and $f, g \in$ F. Writing $c = u + iv$ and using that f, g are real, we get $f(x) = (1 - u)\cos x + v \sin x$, $g(x) = -v \cos x - u \sin x$. For any interval I on which we can use the definition of $f, g \in$ F (and on which $\sin x \neq 0$) we therefore have $F, G \in$ H and $U \in$ U so that $F(z) = (1 - u)z + v\gamma(z)$ and $G(z) = -vz - u\gamma(z)$, where γ is a branch of $\sqrt{1 - z^2}$ on U. We have that

$$\limsup_{|z| \to \infty, z \in U} \frac{|F(z)|}{|z|} = \limsup_{|z| \to \infty, z \in U} \frac{|((1 - u)z + v\gamma(z))|}{|z|} = \sqrt{(1 - u)^2 + v^2}.$$

Since this limit has to be zero we conclude that $u = 1$ and $v = 0$. A similar argument for G gives that $u = v = 0$, a contradiction.