Problemen

Problem Section

Problem (Star) 2008-2/1 Let the continuous function $f_1 : (0,1] \to \mathbf{C}$ be such that

$$\int_0^1 f_1(t)dt$$

exists (and is finite) as an improper Riemann integral. Prove that f_1 has a unique extension to $f\colon \mathbf{R}^+ \to \mathbf{C}$ that is

- continuous on $\mathbf{R}^+$,
- differentiable on $(1,\infty)$ and satisfies the differential-difference equation

$$f'(x) = -\frac{1}{x} f(x-1) \qquad (x > 1). \tag{1}$$

Also, determine

$$\lim_{x\to\infty} x f(x).$$

Finally, show that, if $\int_0^1 f_1(t)dt = f_1(1)$, then the series $\sum_{n=0}^{\infty} n f(n)$ and the integral

$$\int_0^\infty f(t)dt$$

both converge absolutely and have the same value.

Solution We received solutions from Joris Bierkens and J. Arias de Reyna & J. van de Lune. Joris Bierkens will receive the prize.

The following solution is based on the one given by Bierkens.

Define the functions $f_n\colon (n-1,n] \to \mathbf{C}$ inductively, by

$$f_n(x) := f_{n-1}(n-1) - \int_{n-1}^{x} \frac{1}{t} f_{n-1}(t-1)dt,$$

and glue them to a function f on $\mathbf{R}^+$. By the properties of the Riemann integral, this f is continuous on $\mathbf{R}^+$ and differentiable on $(1,\infty)$ and it satisfies the differential-difference equation (1). If $g\colon \mathbf{R}^+ \to \mathbf{C}$ is another function with these properties, then we see that $g'(t) = f'(t)$ on $(1,2]$. From $f(1) = g(1)$ we conclude $f = g$ on $(1,2]$. Repeating this argument it follows that $g = f$ everywhere on $\mathbf{R}^+$.

In order to determine $\lim_{x\to\infty} x f(x)$, note that (1) implies

$$(xf(x))' = f(x) - f(x-1). \tag{2}$$

Therefore

$$\lim_{x\to\infty} x f(x) = f(1) + \int_1^\infty (xf(x))'dx = f_1(1) - \int_0^1 f_1(x)dx,$$

provided that the limit $\lim_{x\to\infty} x f(x)$ exists.

For the last part of the problem, integrate (2) to obtain the recursion

$$\int_n^{n+1} f(t)dt = \int_{n-1}^{n} f(t)dt + (n+1)f(n+1) - nf(n) \qquad (n \geq 1).$$

Now suppose $\int_0^1 f_1(t)dt = f_1(1)$. This recursion implies

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$$\int_{n-1}^{n} f(t)dt = nf(n). \tag{3}$$

We have for $n > 1$

$$\begin{aligned}\int_{n}^{n+1} |f(t)|dx &= \int_{n}^{n+1} \left| f(n) - \int_{n}^{x} \frac{1}{t} f(t-1)dt \right| dx \\ &\leq |f(n)| + \int_{n}^{n+1} \frac{1}{n} \int_{n}^{n+1} |f(t-1)|dtdx \\ &\leq \frac{1}{n} \int_{n-1}^{n} |f(t)|dt + \frac{1}{n} \int_{n}^{n+1} |f(t-1)|dt = \frac{2}{n} \int_{n-1}^{n} |f(t)|dt.\end{aligned}$$

So, by the ratio test, the series

$$\sum_{n=1}^{\infty} \int_{n-1}^{n} |f(t)|dt$$

converges. Since we have

$$\sum_{n=1}^{\infty} |nf(n)| = \sum_{n=1}^{\infty} \left| \int_{n-1}^{n} f(t)dt \right| \leq \int_{0}^{\infty} |f(t)|dt = \sum_{n=1}^{\infty} \int_{n-1}^{n} |f(t)|dt,$$

we conclude that both $\sum_{n=0}^{\infty} nf(n)$ and $\int_0^{\infty} f(t)dt$ converge absolutely, and from (3) it follows that they have the same limit.

Problem (Star) 2008-2/4 Let $p\colon [0,1] \to \mathbf{R}$ be a continuous function with $p(t) \geq 0$ for all $t \in [0,1]$ and $\int_0^1 p(t)dt = 1$. Does the function $f\colon \mathbf{C} \to \mathbf{C}$ given by

$$f(z) = e^z - \int_0^1 p(t)e^{zt}dt$$

have infinitely many zeroes?

Solution We received solutions from R.A. Kortram and J. Arias de Reyna & J. van de Lune. R.A. Kortram will receive the prize.
The following solution is based on the one given by Kortram.
We shall prove that the answer is 'yes'. The function f has a Taylor series expansion given by

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{z^n}{n!}$$

with $a_n = 1 - \int_0^1 t^n p(t)dt$. The coefficients a_n are real and satisfy $0 < a_n < 1$ so for all $r \in \mathbf{R}_{>0}$ we have

$$M_r(f) := \max_{|z|=r} |f(z)| = f(r) < e^r. \tag{4}$$

This shows that f is of order (at most) 1: the order of the entire function f is the infimum of all m such that $f(z) = O(e^{|z|^m})$ as $z \to \infty$.
From now on, assume that f has only finitely many zeroes $z_1, \ldots, z_N$ with multiplicities $e_1, \ldots, e_N$. Hadamard's factorization theorem tells us how an entire function of given order can be expressed as product in terms of its zeroes and leads in our case to

$$f(z) = \phi(z)e^{\lambda z+\mu} \quad \text{with} \quad \phi(z) = \prod_{j=1}^{N}(z-z_j)^{e_j}$$

for certain $\lambda, \mu \in \mathbf{C}$. Since the Taylor coefficients of f are real, we have $\phi(z) \in \mathbf{R}[z]$, $\lambda \in \mathbf{R}$ and $e^{\mu} \in \mathbf{R}$ and hence there is a real number c with

$$f(z) = c\phi(z)e^{\lambda z}.$$

Now put $g(z) = \int_0^1 p(t)e^{zt}dt = e^z - f(z)$. We have

$$g(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} \quad \text{with} \quad b_n = \int_0^1 t^n p(t)dt > 0.$$

Hence for $r \in \mathbf{R}_{>0}$ we have

$$M_r(g) := \max_{|z|=r} |g(z)| = g(r) = e^r - f(r) = e^r - c\phi(r)e^{\lambda r}.$$

The fact that $f(0) = 0$ implies $\deg(\phi) \geq 1$; combining this with (4) we get $\lambda < 1$. So there is an $R \in \mathbf{R}$ such that for all $r > R$ we have $M_r(g) > e^r/2$.
Choose $\varepsilon < 1/4$ and $\delta \in [0,1)$ with $\int_\delta^1 p(t)dt < \varepsilon$. Then also $\int_\delta^1 t^n p(t)dt < \varepsilon$. Choose K with $\delta^K < \varepsilon$. For $n \geq K$ we have

$$\int_0^\delta t^n p(t)dt \leq \delta^n \int_0^\delta p(t)dt \leq \delta^n \int_0^1 p(t)dt < \varepsilon$$

and thus $b_n < 2\varepsilon$.
For $r \geq R$ we get the following inequality:

$$e^r/2 < M_r(g) = g(r) < \sum_{n=0}^{K-1} b_n \frac{r^n}{n!} + 2\varepsilon \sum_{n=K}^{\infty} \frac{r^n}{n!} < \sum_{n=0}^{K-1} b_n \frac{r^n}{n!} + 2\varepsilon \cdot e^r,$$

which is a contradiction for large r.