Problemen/UWC

Universitaire Wiskunde Competitie

Edition 2005/3
For Session 2005/3 of the Universitaire Wiskunde Competitie we received submissions from DESDA (Nijmegen), Ruud Jeurissen, the team A.P.M. Kupers en J.W.T. Konter, and Jaap Spies.

Problem 2005/3-A In what follows f, g are two continuous functions.
1) Determine $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that $f \circ g(x) = x + 1$ and $g \circ f(x) = x - 1$.
2) Determine $f : \mathbf{R}^+ \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that $f \circ g(x) = x + 1$ and $g \circ f(x) = 2x$.
As usual, the symbol '$\circ$' denotes the composition of functions and $\mathbf{R}^+$ the set of all strict positive real numbers.

Solution This problem was solved by DESDA (Nijmegen), Ruud Jeurissen and the team A.P.M. Kupers en J.W.T. Konter. The solution below is based on that of Ruud Jeurissen.
1) $f(x) + 1 = f \circ g \circ f(x) = f(x - 1)$, so there is an a such that $f(x) = -x + a$. $g(x) - 1 = g \circ f \circ g(x) = g(x + 1)$, so there is a b such that $g(x) = -x + b$. Then $f \circ g(x) = f(-x + b) = x - b + a$, so we must have $a - b = 1$, in which case $g \circ f(x) = g(-x + a) = x - a + b = x - 1$, as desired.
2) Since $f \circ g$ is defined, g can only take positive values. For $x > 0$ we have $f(x) + 1 = f \circ g \circ f(x) = f(2x)$, so there is a b such that $f(x) = {}^2\log x + b$. For all x we have $2g(x) = g \circ f \circ g(x) = g(x + 1)$, so there is an a such that $g(x) = 2^{x+a}$. Then $f \circ g(x) = f(2^{x+a}) = x + a + b$, so we must have $a + b = 1$, in which case $g \circ f(x) = g({}^2\log x + b) = x \cdot 2^{b+a} = 2x$, as desired.

Problem Generalisation The team A.P.M. Kupers en J.W.T. Konter considered the following generalisation of part 2):
Determine f and g such that $f \circ g(x) = x + 1$ and $g \circ f(x) = ax + b$. They found that f and g must satisfy

$$f(x) = \frac{Log\left(\frac{(ax+b)(a-1)+b}{b+ca-c}\right)}{Log(a)}, \qquad g(k) = \frac{(a^k - 1)b}{a - 1} + c \cdot a^k$$

Problem 2005/3-B
1. Let G be a group and suppose that the maps $f, g : G \to G$ with $f(x) = x^3$ and $g(x) = x^5$ are both homomorphisms. Show that G is Abelian.
2. In the previous exercise, by which pairs (m, n) can $(3, 5)$ be replaced if we still want to be able to prove that G is Abelian.

Solution This problem was solved by Jaap Spies and the team A.P.M. Kupers en J.W.T. Konter. The solution below is based on that of Jaap Spies.
By assumption we have $(ab)^5 = a^5b^5$ for all $a, b \in G$. We easily see that $(ba)^4 = a^4b^4$. Likewise, $(ab)^3 = a^3b^3$ for all $a, b \in G$ and hence $(ba)^2 = a^2b^2$. So $(a^2b^2)^2 = a^4b^4$ and $b^2a^2 = a^2b^2$. Hence squares commute in G. Now $a^4b^4 = b^4a^4 = (ba)^4$ and so $b^3a^3 = (ab)^3 = a^3b^3$. Hence cubes also commute in G. In the solution of Problem 2003/4-B of the UWC, it was proved that in this case G is Abelian.

Problem 2005/3-C For $s > 1$ define

$$\psi_1(s) = \prod_p (1 - \frac{1}{p^s})^{-1} \text{ with } p \text{ over all primes } \equiv 1 \bmod 4$$

and

$$\psi_3(s) = \prod_q (1 - \frac{1}{q^s})^{-1} \text{ with } q \text{ over all primes } \equiv 3 \bmod 4.$$

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Describe how $\lim_{s\downarrow 1} \frac{\psi_3(s)}{\psi_1(s)}$ can be computed to 'any' degree of (high) accuracy (precision). (The use of an algebra-package is permitted.)

Solution This problem has been solved by the team A.P.M. Kupers en J.W.T. Konter. The solution below is based on their solution. It has been shortened for publication; the complete text, with references and calculations, can be found on the UWC website.

The relation between ψ_1, ψ_3 and the ζ-function
The Riemann-Zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It can also be written as a product over the prime numbers:

$$\zeta(s) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}$$

where p_n is the n-th prime number. As all odd prime numbers are either 1 or 3 modulo 4, we can rewrite this as:

$$\zeta(s) = \frac{1}{\left(1 - \frac{1}{2^s}\right)} \psi_1(s)\psi_3(s).$$

Is the ratio $\psi_3(1)/\psi_1(1)$ a real number?
Both $\psi_3(1)$ and $\psi_1(1)$ are infinite. Is the ratio $\psi_3(1)/\psi_1(1)$ a real number? Using the number theoretic character χ_4, we can prove that this ratio is equal to $(4/\pi)\psi_3(2)$, which is indeed a real number.

The ratio $\psi_1(1)/\psi_3(1)$
Likewise, we can show that $\psi_1(1)/\psi_3(1)$ is equal to $(2/\pi)\psi_1(2)$.

The idea behind the approximation
If we divide the expression that we found for $\psi_3(1)/\psi_1(1)$ by the one we found for $\psi_1(1)/\psi_3(1)$, we obtain

$$\left(\frac{\psi_3(1)}{\psi_1(1)}\right)^2 = 2\frac{\psi_3(2)}{\psi_1(2)}, \qquad \frac{\psi_3(1)}{\psi_1(1)} = \sqrt{2\frac{\psi_3(2)}{\psi_1(2)}}.$$

The idea behind the approximation is that $\psi_3(2)/\psi_1(2)$ can again be written as the square root of a constant times $\psi_3(4)/\psi_1(4)$, etc. This turns out to be correct.

Oplossingen

The Dirichlet L-Series

The Dirichlet L-series is defined as

$$L_k(s,\chi_k) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s}.$$

We can show that

$$L_4(s,\chi_4) = \prod_n \left(1 - \frac{\chi_4(p_n)}{p_n^s}\right)^{-1}$$

Now L_4 is equal to the Dirichlet β-function, that is,

$$L_4(s,\chi_4) = \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

In particular, the sum in the β-function is over all integers, and not just the primes.

The ratios $\frac{\psi_3(2^n)}{\psi_1(2^n)}$ and $\frac{\psi_1(2^n)}{\psi_3(2^n)}$

If we divide the ratio $\psi_3(2^n)/\psi_1(2^n)$ for n$\in \mathbf{N}\cup\{0\}$ by $\psi_3(2^{n+1})$, then through calculations as above, we obtain

$$\frac{\psi_3(2^n)}{\psi_3(2^{n+1})\psi_1(2^n)} = \prod_p \left(1 - \frac{1}{p^{2^n}}\right) \prod_q \left(1 + \frac{1}{q^{2^n}}\right).$$

The result is equal to $[L_4(2^n,\chi)]^{-1}$, which in turn is equal to $\beta(2^n)^{-1}$. For $\psi_3(2^n)/\psi_1(2^n)$ we therefore find the expression $\psi_3(2^{n+1})/\beta\,(2^n)$. Let us now consider $\psi_1(2^n)/\psi_3(2^n)$:

$$\frac{\psi_1(2^n)}{\psi_1(2^{n+1})\,\psi_3(2^n)} = \prod_q \left(1 - \frac{1}{q^{2^n}}\right) \prod_p \left(1 + \frac{1}{p^{2^n}}\right).$$

We can again recognise the Dirichlet L-series in here, and after some calculation, we find

$$\frac{\psi_1(2^n)}{\psi_3(2^n)} = \frac{\beta\,(2^n)}{\left(1 - \frac{1}{2^{2^{n+1}}}\right)\zeta\,(2^{n+1})}\psi_1\left(2^{n+1}\right)$$

A recursive formula

If we now divide the expression we found for $\psi_3(2^n)/\psi_1(2^n)$ by the one we found for $\psi_1(2^n)/\psi_3(2^n)$, we find

$$\left(\frac{\psi_3(2^n)}{\psi_1(2^n)}\right)^2 = \frac{\left(1 - \frac{1}{2^{2^{n+1}}}\right)\zeta\,(2^{n+1})}{\beta\,(2^n)^2}\frac{\psi_3(2^{n+1})}{\psi_1(2^{n+1})}$$

$$\frac{\psi_3(2^n)}{\psi_1(2^n)} = \sqrt{\frac{\left(1 - \frac{1}{2^{2^{n+1}}}\right)\zeta\,(2^{n+1})}{\beta\,(2^n)^2}\frac{\psi_3(2^{n+1})}{\psi_1(2^{n+1})}}.$$

This is recursive formula that allows us to deduce the ratio $\psi_3(2^n)/\psi_1(2^n)$ from the ratio $\psi_3(2^{n+1})/\psi_1(2^{n+1})$. Each time this recursive formula is applied to a ratio $\psi_3(s)/\psi_1(s)$, s is divided by 2. This way we can approximate the ratio $\psi_3(1)/\psi_1(1)$. We do not need to use sums or products over primes because both ζ and β can be approximated without these, for example using an algebra-package such as *Mathematica* or *Matlab*. Consider the following limits:

$$\lim_{s\to\infty}\psi_1(s) = \lim_{s\to\infty}\prod_p \frac{1}{\left(1-\frac{1}{p^s}\right)} = 1, \qquad \lim_{s\to\infty}\psi_3(s) = \lim_{s\to\infty}\prod_q \frac{1}{\left(1-\frac{1}{q^s}\right)} = 1$$

Consequently the following limit also tends to 1:

$$\lim_{s\to\infty}\frac{\psi_3(s)}{\psi_1(s)} = 1$$

We can therefore approximate $\psi_3(1)/\psi_1(1)$ as follows:

- Choose a positive integer n.
- Approximate $\frac{\psi_3(2^n)}{\psi_1(2^n)}$ by supposing that $\frac{\psi_3(2^n)}{\psi_1(2^n)} = 1$.
- Use the recursive formula a number of times to obtain $\frac{\psi_3(1)}{\psi_1(1)}$.

The ratio $\psi_3(1)/\psi_3(1)$ can thus be approximated by the following limit:

$$\lim_{n\to\infty} \sqrt{\frac{\left(1-\frac{1}{2^2}\right)\zeta(2)}{\beta(1)^2}\sqrt{\frac{\left(1-\frac{1}{2^4}\right)\zeta(4)}{\beta(2)^2}\sqrt{\frac{\left(1-\frac{1}{2^8}\right)\zeta(8)}{\beta(4)^2}\cdots\sqrt{\frac{\left(1-\frac{1}{2^{2*2^n}}\right)\zeta\left(2^{n+1}\right)}{\beta\left(2^n\right)^2}*1}}}}$$

The precision of the approximation depends on three factors:

- The number n: the larger n is, the better the approximation.
- The precision used in the approximation of the ζ and β-functions: the more precise these are, the better the approximation of the ratio. Nowadays, with algebra-packages such as *Mathematica* and *Maple*, this is no problem.
- The precision used in calculation the square root: don't forget that the square root is also an approximation. *Mathematica* and *Maple* have no problem with this.

A trial approximation with Mathematica

The following functions approximate $\psi_3(s)$ and $\psi_1(s)$ by only considering the first n primes.

```
p1[s\_, n\_] :=
 Module[{x = 1}, {pmo1 =
  DeleteCases[
   Table[If[Mod[Prime[i], 4] == 1, Prime[i], 0],
                  {i, 1, n}], 0];
   Product[(1 - pmo1[[i]]$\hat{}$(-s))$\hat{}$-1,
                  {i, 1, Length[pmo1]}]}][[1]]
```

```
p3[s\_, n\_] :=
 Module[{x = 1}, {pmo3 =
  DeleteCases[
   Table[If[Mod[Prime[i], 4] == 3, Prime[i], 0],
                  {i, 1, n}], 0];
   Product[(1 - pmo3[[i]]$\hat{}$(-s))$\hat{}$-1,
                  {i, 1, Length[pmo3]}]}][[1]]
```

Let us consider the ratio for the first 10000 primes. Timing[] determines the time it takes *Mathematica* to compute this.

```
p3p1 = Timing[N[p3[1, 100000]/p1[1, 100000], 20]]
37.594 Second, 1.4871655814206811459
```

Mathematica takes about 37,5 seconds to do this.

The β-function is a sum, as is the ζ-function, but it is not standard in *Mathematica*. We must first define it:

```
DB[x\_, k\_] := Sum[(-1)$\hat{}$n/(2n + 1)$\hat{}$(x), {n, 0, k}]
```

```
BLIM[n\_] :=
 Module[{x = n, expr = 1},
  While[x != -1,
   expr = Sqrt[(1 - 2$\hat{}$(-2$\hat{}$(x + 1)))*
   Zeta[2$\hat{}$(x + 1)]/(DB[2$\hat{}$x,
      $\setminus$[Infinity]])$\hat{}$2*expr]; x = x - 1]; expr]
```

Let us first make a table with the approximations for n = $\{1, 2, 3, 4, 5\}$, then with the differences between the approximations and the value computed above with 10000 primes.

```
approximation = Table[Timing[N[BLIM[i], 20]], {i, 1, 5}]
```

```
{{0.032 Second, 1.4830557664224863250}, {0.046 Second,
 1.4872121328716638225}, {0.094 Second,
```

```
 1.4872400244256083148}, {0.125 Second,
 1.4872400265843418256}, {0.188 Second, 1.4872400265843418507}}
```

Even the best approximation only takes 0,2 seconds. But how large is the deviation?

```
Table[approximation[[i]][[2]] - p3p1, {i, 1, 5}]
```

```
-0.0041098149981948210, 0.0000465514509826766,
0.0000744430049271689, \ 0.0000744451636606797,
0.0000744451636607048
```

This approximation is so much better than the brute-force method with the prime numbers that even for $n = 2$ it is already very close. And it is almost 150 times faster.

Possible generalisations

For this case we worked with the number theoretic character χ_4 and the corresponding L-series. It is possible to generalise the solution to other number theoretic characters. For example, for the functions ψ_5 and ψ_1, for which we would use the character χ_6, the following holds:

$$\psi_1(s) = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

where p runs over all primes that are 1 modulo 6.

$$\psi_5(s) = \prod_q \frac{1}{\left(1 - \frac{1}{q^s}\right)}$$

where q runs over all primes that are 5 modulo 6.

We can therefore approximate the ratio $\psi_5(1)/\psi_1(1)$ as follows:

- Choose a positive integer n.
- Approximate $\frac{\psi_5(2^n)}{\psi_1(2^n)}$ by supposing that $\frac{\psi_5(2^n)}{\psi_1(2^n)} = 1$
- Use the following recursive formula a number of times to obtain $\frac{\psi_5(1)}{\psi_1(1)}$:

$$\frac{\psi_5\left(2^{n-1}\right)}{\psi_1\left(2^{n-1}\right)} = \sqrt{\frac{\left(1 - \frac{1}{2^{2^n}}\right)\left(1 - \frac{1}{3^{2^n}}\right)\zeta\left(2^n\right)}{L_6\left(2^{n-1}, \chi_6\right)^2} \frac{\psi_5\left(2^n\right)}{\psi_1\left(2^n\right)}}$$

The extra term with 3 is due to the fact that the character χ_6 excludes 2 and 3, which are not coprime with 6, while χ_4 only excludes 2.

1 Problem 2005/3 C

(Proposed by Jan van de Lune)

For $s > 1$ define

$$\Phi_1(s) = \prod_p (1 - \frac{1}{p^s})^{-1} \text{ with } p \text{ over all primes } \equiv 1 \bmod 4$$

and

$$\Phi_3(s) = \prod_q (1 - \frac{1}{q^s})^{-1} \text{ with } q \text{ over all primes } \equiv 3 \bmod 4.$$

Describe how $\lim_{s\downarrow 1} \frac{\Phi_3(s)}{\Phi_1(s)}$ can be computed to "any" degree of (high) accuracy (precision). (The use of an algebra-package is permitted.)

2 Solution

The editors got solutions from the team A.P.M. Kupers en J.W.T. Konter. The solution below is based on their solution.

Definiëer voor s > 1

$$\psi 1[s] = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)} \quad (1)$$

voor p alle priemgetallen die modulo 4 gelijk zijn aan 1

$$\psi 3[s] = \prod_q \frac{1}{\left(1 - \frac{1}{q^s}\right)} \quad (2)$$

voor q alle priemgetallen die modulo 4 gelijk zijn aan 3
Beschrijf een manier om $\lim_{s\downarrow 0} \frac{\psi 3(s)}{\psi 1(s)}$met een willekeurige precisie.

2.1 Het verband tussen $\psi 1$, $\psi 3$ en de ζ-functie

De Riemann-Zeta functie is gedefiniëerd als:

$$\zeta[s] = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (3)$$

Het is ook mogelijk om de Zeta-functie te schrijven als een product van de priemgetallen (http://mathworld.wolfram.com/RiemannZetaFunction.html, 42):

$$\zeta[s] = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1} \quad (4)$$

waarbij p_n het n-de priemgetal is.
Alle priemgetallen behalve 2 zijn oneven. Alle oneven getallen modulo 4 zijn 1 of 3. Daarom is de Zeta-functie ook te schrijven als:

$$\zeta[s] = \frac{1}{(1 - \frac{1}{2^s})} \psi 1[s] \psi 3[s] \quad (5)$$

2.2 Is de verhouding $\psi_3[1]/\psi_1[1]$ een reëel getal?

Zowel $\psi 3[1]$ als $\psi 1[1]$ is oneindig. Is de verhouding $\psi 3[1]/\psi 1[1]$ wel een reëel getal? Laten we deze verhouding uitschrijven volgens uitdrukkingen 1 en 2:

$$\frac{\psi 3[1]}{\psi 1[1]} = \frac{\prod_q \frac{1}{\left(1 - \frac{1}{q}\right)}}{\prod_p \frac{1}{\left(1 - \frac{1}{p}\right)}} = \frac{\prod_p \left(1 - \frac{1}{p}\right)}{\prod_q \left(1 - \frac{1}{q}\right)} \quad (6)$$

Vermenigvuldig deze uitdrukking met $1/\psi 3[2]$:

$$\frac{\psi 3[1]}{\psi 3[2] \psi 1[1]} = \frac{\prod_q \frac{1}{\left(1 - \frac{1}{q}\right)}}{\prod_p \frac{1}{\left(1 - \frac{1}{p}\right)} \prod_q \frac{1}{\left(1 - \frac{1}{q^2}\right)}} = \frac{\prod_p \left(1 - \frac{1}{p}\right) \prod_q \left(1 - \frac{1}{q^2}\right)}{\prod_q \left(1 - \frac{1}{q}\right)}$$

Je kunt elk element van het oneindige product $\psi 3[2]$ uitschrijven door het kwadraat:

$$\left(1 - \frac{1}{q^2}\right) = \left(1 + \frac{1}{q}\right)\left(1 - \frac{1}{q}\right)$$

Dit invullen geeft:

$$\frac{\psi 3[1]}{\psi 3[2] \psi 1[1]} = \frac{\prod_p \left(1 - \frac{1}{p}\right) \prod_q \left(1 + \frac{1}{q}\right)\left(1 - \frac{1}{q}\right)}{\prod_q \left(1 - \frac{1}{q}\right)} = \prod_p \left(1 - \frac{1}{p}\right) \prod_q \left(1 + \frac{1}{q}\right) \quad (7)$$

We gebruiken nu het number theoretic character χ_4 gedefiniëerd als (http://mathworld.wolfram.com/Nu

$$\chi_4[n] \equiv \{ \begin{matrix} +1 \text{ als } n \bmod 4 = 1 \\ -1 \text{ als } n \bmod 4 = 3 \\ 0 \text{ anders} \end{matrix}$$

Volgens de vraagstellingen waren alle priemgetallen p modulo 4 gelijk aan 1, q modulo 4 gelijk aan 3. χ laat ons (7) schrijven als:

$$\frac{\psi 3[1]}{\psi 3[2]\psi 1[1]} = \prod_{k=2} \left(1 - \frac{\chi_4[k]}{k}\right) \ (8)$$

(met k_n het n-de priemgetallen)
Volgens uitdrukking 17 op http://mathworld.wolfram.com/PrimeProducts.html, is dit gelijk aan (in een latere sectie bewijs ik dit ook):

$$\frac{\psi 3[1]}{\psi 3[2]\psi 1[1]} = \prod_{k=2} \left(1 - \frac{\chi_4[k]}{k}\right) = \left(\prod_{k=2} \left(1 - \frac{\chi_4[k]}{k}\right)^{-1}\right)^{-1} = \left(\frac{\pi}{4}\right)^{-1} = \frac{4}{\pi}$$

We kunnen de verhouding $\psi 3[1]/\psi 1[1]$ dus schrijven als:

$$\frac{\psi 3[1]}{\psi 1[1]} = \frac{4}{\pi}\psi 3[2] \ (9)$$

Omdat $\psi 3[2]$ gewoon een reëel getal is, is de verhouding ook een reëel getal.

2.3 De verhouding $\psi_1[1]/\psi_3[1]$

Omgekeerd is het ook mogelijk:

$$\frac{\psi 1[1]}{\psi 3[1]} = \frac{\prod_p \frac{1}{\left(1-\frac{1}{p}\right)}}{\prod_q \frac{1}{\left(1-\frac{1}{q}\right)}} = \frac{\prod_q \left(1 - \frac{1}{q}\right)}{\prod_p \left(1 - \frac{1}{p}\right)} \ (10)$$

Op dezelfde manier krijgen we met vermenigvuldiging:

$$\frac{\psi 1[1]}{\psi 1[2]\psi 3[1]} = \frac{\prod_q \left(1 - \frac{1}{q}\right) \prod_p \left(1 + \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)}{\prod_p \left(1 - \frac{1}{p}\right)} = \prod_q \left(1 - \frac{1}{q}\right) \prod_p \left(1 + \frac{1}{p}\right) = \prod_{k=2} \left(1 + \frac{\chi_4[k]}{k}\right)$$

Dit product kunnen we gelukkig omschrijven naar het al bekende product 8:

$$\prod_{k=2}\left(1+\frac{\chi_4[k]}{k}\right)=\frac{\prod_{k=2}\left(1-\frac{1}{k^2}\right)}{\prod_{k=2}\left(1-\frac{\chi_4[k]}{k}\right)}$$

Herinner je je definitie van de ζ-functie nog?

$$\prod_{k=2}\left(1-\frac{1}{k^2}\right)=\left(\left(1-\frac{1}{2^2}\right)\zeta[2]\right)^{-1}$$

waarbij de factor met 2 compenseert voor het product dat bij 2 i.p.v. 1 begint. Dus:

$$\prod_{k=2}\left(1+\frac{\chi_4[k]}{k}\right)=\frac{\pi}{4}\frac{1}{\left(1-\frac{1}{4}\right)\zeta[2]}$$

Omdat $\zeta[2]$ gelijk is aan $\pi^2/6$ (http://mathworld.wolfram.com/RiemannZetaFunction.html), geldt:

$$\prod_{k=2}\left(1+\frac{\chi_4[k]}{k}\right)=\frac{\pi}{4}\frac{6}{\pi^2}\frac{4}{3}=\frac{2}{\pi}$$

De verhouding $\psi1[1]/\psi3[1]$ is dus:

$$\frac{\psi1[1]}{\psi3[1]}=\frac{2}{\pi}\psi1[2]\ (11)$$

2.4 Het idee voor een benaderingsmethode

We hebben dus gevonden:

$$\frac{\psi1[1]}{\psi3[1]}=\frac{2}{\pi}\psi1[2]$$

$$\frac{\psi3[1]}{\psi1[1]}=\frac{4}{\pi}\psi3[2]$$

De eerste uitdrukking tot de -1de macht doen en vermenigvuldingen levert:

$$\left(\frac{\psi3[1]}{\psi1[1]}\right)^2=2\frac{\psi3[2]}{\psi1[2]}$$

$$\frac{\psi3[1]}{\psi1[1]}=\sqrt{2\frac{\psi3[2]}{\psi1[2]}}$$

Het idee van mijn benadering is dat $\psi3[2]/\psi1[2]$ weer geschreven kan worden als de wortel van een constante maal $\psi3[4]/\psi1[4]$, etc. Dit blijkt inderdaad zo te zijn.

2.5 De Dirichlet L-Series

Er is een series, de Dirichtlet L-Series (http://mathworld.wolfram.com/DirichletL-Series.html) genaamd, die gedefiniëerd is als:

$$L_k\left(s,\chi_k\right)=\sum_{n=1}^{\infty}\frac{\chi_k[n]}{n^s}$$

Wij zijn geïnteresseerd in L_4, omdat die ook in onze verhouding voorkwam.
Dit is, net als de ζ-functie, ook te schrijven als een product van priemgetallen. We gebruiken dezelfde methode als bij de ζ-functie om dit aan te tonen.

$$\left(1-\frac{\chi_4[3]}{3^s}\right)L_4\left(s,\chi_4\right)=\left(1+\frac{1}{3^s}\right)\sum_{n=1}^{\infty}\frac{\chi_4[n]}{n^s}=\left(1+\frac{1}{3^s}\right)\left(1-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}+...\right)=$$

$$=\left(1-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}+...\right)+\left(\frac{1}{3^s}-\frac{1}{9^s}+\frac{1}{15^s}-\frac{1}{21^s}+...\right)=1+\frac{1}{5^s}-\frac{1}{7^s}-\frac{1}{11^s}+...$$

De factor $(1-\frac{\chi_4[3]}{3^s})$ haalt dus alle getallen met een priemfactor 3 eruit. $(1-\frac{\chi_4[5]}{5^s})$ zal alles met een priemfactor 5 eruit halen, etc. We kunnen dus stellen dat:

$$\prod_k\left(1-\frac{\chi_4[k]}{k^s}\right)L_4\left(s,\chi_4\right)=1$$

$$L_4\left(s,\chi_4\right)=\prod_k\left(1-\frac{\chi_4[k]}{k^s}\right)^{-1}$$

Nu blijkt (ik ga dit niet bewijzen, dat is te moeilijk) dat L_4 gelijk is aan L_{-4} en dat L_{-4} gelijk is aan de Dirichtlet β-functie (zie http://mathworld.wolfram.com/DirichletL-Series.html). Deze β-functie (http://mathworld.wolfram.com/DirichletBetaFunction.html) is gedefiniëerd als:

$$L_4\left(s,\chi_4\right)=\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

Er komen dus geen priemgetallen in de β-functie voor.

2.6 De verhoudingen $\frac{\psi_3[2^n]}{\psi_1[2^n]}$ en $\frac{\psi_1[2^n]}{\psi_3[2^n]}$

De verhouding $\psi 3[2^n]/\ \psi 1[2^n]$ voor n$\in \mathbb{N}\cup\{0\}$ is gelijk aan:

$$\frac{\psi 3\left[2^n\right]}{\psi 1\left[2^n\right]}=\frac{\prod_q \frac{1}{\left(1-\frac{1}{q^{2^n}}\right)}}{\prod_p \frac{1}{\left(1-\frac{1}{p^{2^n}}\right)}}=\frac{\prod_p\left(1-\frac{1}{p^{2^n}}\right)}{\prod_q\left(1-\frac{1}{q^{2^n}}\right)}$$

Vermenigvuldigen met $1/\psi 3[2^{n+1}]$ geeft:

$$\frac{\psi 3\left[2^n\right]}{\psi 3\left[2^{n+1}\right]\psi 1\left[2^n\right]}=\frac{\prod_p\left(1-\frac{1}{p^{2^n}}\right)\prod_q\left(1+\frac{1}{q^{2^n}}\right)\left(1-\frac{1}{q^{2^n}}\right)}{\prod_q\left(1-\frac{1}{q^{2^n}}\right)}=\prod_p\left(1-\frac{1}{p^{2^n}}\right)\prod_q\left(1+\frac{1}{q^{2^n}}\right)$$

Want je kunt hier immers weer $\psi 3[2^{n+1}]$ uitsplitsen. De uitkomst lijkt op $L_4[2^n,\chi]$

$$\frac{\psi 3\left[2^n\right]}{\psi 3\left[2^{n+1}\right]\psi 1\left[2^n\right]}=\prod_p\left(1-\frac{1}{p^{2^n}}\right)\prod_q\left(1+\frac{1}{q^{2^n}}\right)=\prod_{k=2}\left(1-\frac{\chi_4[k]}{k^{2^n}}\right)=\frac{1}{L_4\left[2^n,\chi\right]}=\frac{1}{\beta\left[2^n\right]}$$

We vinden dus voor $\psi 3[2^n]/\psi 1[2^n]$ de uitdrukking:

$$\frac{\psi 3\left[2^n\right]}{\psi 1\left[2^n\right]}=\frac{\psi 3\left[2^{n+1}\right]}{\beta\left[2^n\right]}$$

Nu gaan we de verhouding $\psi 1[2^n]/\ \psi 3[2^n]$ uitrekenen:

$$\frac{\psi 1\left[2^n\right]}{\psi 3\left[2^n\right]}=\frac{\prod_p \frac{1}{\left(1-\frac{1}{p^{2^n}}\right)}}{\prod_q \frac{1}{\left(1-\frac{1}{q^{2^n}}\right)}}=\frac{\prod_q\left(1-\frac{1}{q^{2^n}}\right)}{\prod_p\left(1-\frac{1}{p^{2^n}}\right)}$$

$$\frac{\psi 1\left[2^n\right]}{\psi 1\left[2^{n+1}\right]\psi 3\left[2^n\right]}=\frac{\prod_q\left(1-\frac{1}{q^{2^n}}\right)\prod_p\left(1+\frac{1}{p^{2^n}}\right)\left(1-\frac{1}{p^{2^n}}\right)}{\prod_p\left(1-\frac{1}{p^{2^n}}\right)}=\prod_q\left(1-\frac{1}{q^{2^n}}\right)\prod_p\left(1+\frac{1}{p^{2^n}}\right)$$

Hier kunnen we weer de Dirichlet L-Series in vinden:

$$\frac{\psi 1\left[2^n\right]}{\psi 1\left[2^{n+1}\right]\psi 3\left[2^n\right]}=\prod_{k=2}\left(1+\frac{\chi_4[k]}{k^{2^n}}\right)=\frac{\prod_{k=2}\left(1-\frac{1}{\left(k^{2^n}\right)^2}\right)}{\prod_{k=2}\left(1-\frac{\chi_4[k]}{k^{2^n}}\right)}=$$

$$=\frac{\prod_{k=2}\left(1-\frac{1}{k^{2^{n+1}}}\right)}{\prod_{k=2}\left(1-\frac{\chi_4[k]}{k^{2^n}}\right)}=\frac{L_4\left[2^n,\chi_k\right]}{\left(1-\frac{1}{2^{2^{n+1}}}\right)\zeta\left[2^{n+1}\right]}=\frac{\beta\left[2^n\right]}{\left(1-\frac{1}{2^{2^{n+1}}}\right)\zeta\left[2^{n+1}\right]}$$

We vinden dus voor $\psi1[2^n]/\ \psi3[2^n]$ de uitdrukking:

$$\frac{\psi1\left[2^n\right]}{\psi3\left[2^n\right]}=\frac{\beta\left[2^n\right]}{\left(1-\frac{1}{2^{2^{n+1}}}\right)\zeta\left[2^{n+1}\right]}\psi1\left[2^{n+1}\right]$$

2.7 Een recursieve formule

De volgende uitdrukkingen hebben we net afgeleid:

$$\frac{\psi3\left[2^n\right]}{\psi1\left[2^n\right]}=\frac{\psi3\left[2^{n+1}\right]}{\beta\left[2^n\right]}$$

$$\frac{\psi1\left[2^n\right]}{\psi3\left[2^n\right]}=\frac{\beta\left[2^n\right]}{\left(1-\frac{1}{2^{2^{n+1}}}\right)\zeta\left[2^{n+1}\right]}\psi1\left[2^{n+1}\right]$$

De tweede tot de -1de macht doen en vermenigvuldigen levert:

$$\left(\frac{\psi3\left[2^n\right]}{\psi1\left[2^n\right]}\right)^2=\frac{\left(1-\frac{1}{2^{2^{n+1}}}\right)\zeta\left[2^{n+1}\right]}{\beta\left[2^n\right]^2}\frac{\psi3\left[2^{n+1}\right]}{\psi1\left[2^{n+1}\right]}$$

$$\frac{\psi3\left[2^n\right]}{\psi1\left[2^n\right]}=\sqrt{\frac{\left(1-\frac{1}{2^{2^{n+1}}}\right)\zeta\left[2^{n+1}\right]}{\beta\left[2^n\right]^2}\frac{\psi3\left[2^{n+1}\right]}{\psi1\left[2^{n+1}\right]}}$$

Dit is een recursieve formule waarmee je uit de verhouding $\psi3[2^{n+1}]/\psi1[2^{n+1}]$ de verhouding $\psi3[2^n]/\psi1[2^n]$ kunt bepalen. Telkens als je deze recursieve formule op een verhouding $\psi3[1]/\psi1[1]$ toepast, wordt s gedeeld door 2. Zo benader je dus steeds meer de gezochte verhouding $\psi3[1]/\ \psi1[1]$. Er zijn geen priemgetallen meer nodig om de limiet te bepalen, want zowel ζ en β kunnen benaderd worden zonder priemgetallen (dit kan perfect in een algebra-package als *Mathematica* of Matlab):

$$\zeta[s]=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

Bekijk de volgende limieten:

$$\lim_{s\to\infty}\psi1[s]=\lim_{s\to\infty}\prod_p\frac{1}{\left(1-\frac{1}{p^s}\right)}=1$$

$$\lim_{s\to\infty} \psi3[s] = \lim_{s\to\infty} \prod_q \frac{1}{\left(1-\frac{1}{q^s}\right)} = 1$$

Dat betekent dat ook de volgende limiet naar 1 gaat:

$$\lim_{s\to\infty} \frac{\psi3[s]}{\psi1[s]} = 1$$

Want de termen 1-1/p^s en 1-1/q^s naderen 1 steeds meer, naarmate s groter wordt. Het is dus mogelijk om $\psi3[1]/\psi1[1]$ als volgt te benaderen:

1. Kies een natuurlijk getal n.
2. Benader $\frac{\psi3[2^n]}{\psi1[2^n]}$ door te stellen dat $\frac{\psi3[2^n]}{\psi1[2^n]} = 1$.
3. Gebruik de recursieve meerdere malen om uiteindelijk op $\frac{\psi3[1]}{\psi1[1]}$ uit te komen.

De precisie van je benadering hangt af van drie dingen:

1. Het getal n: Hoe groter je n kiest, hoe beter de benadering zal zijn.
2. De precisie waarme we de ζ en β-functies benadert: Hoe beter die precisie is, hoe benadering je benadering van de verhouding zal zijn. Tegenwoordig, met algebra-programma's als Mathematica en Maple, is dit geen probleem meer.
3. De precisie waarmee je de wortel uitrekent: Je moet niet vergeten dat de wortel ook een benadering is. Mathematica en Maple hebben hier geen problemen mee.

Je berekent de limiet perfect met:

$$\frac{\psi3[1]}{\psi3[1]} = \lim_{n\to\infty} \sqrt{\frac{\left(1-\frac{1}{2^2}\right)\zeta[2]}{\beta[1]^2}\sqrt{\frac{\left(1-\frac{1}{2^4}\right)\zeta[4]}{\beta[2]^2}\sqrt{\frac{\left(1-\frac{1}{2^8}\right)\zeta[8]}{\beta[4]^2}\cdots\sqrt{\frac{\left(1-\frac{1}{2^{2*2^n}}\right)\zeta\left[2^{n+1}\right]}{\beta\left[2^n\right]^2}*1}}}}$$

2.8 Een test van de benadering met *Mathematica*

De volgende twee functies benaderen ψ3[s] ψ1[s] door alleen de eerste n priemgetallen te bekijken:

```
p1[s_, n_] :=
  Module[{x = 1}, {pmo1 =
          DeleteCases[
            Table[If[Mod[Prime[i], 4] == 1, Prime[i], 0], {i, 1, n}], 0];
        Product[(1 - pmo1[[i]]^(-s))^-1, {i, 1, Length[pmo1]}]}][[1]]

p3[s_, n_] :=
  Module[{x = 1}, {pmo3 =
          DeleteCases[
            Table[If[Mod[Prime[i], 4] == 3, Prime[i], 0], {i, 1, n}], 0];
        Product[(1 - pmo3[[i]]^(-s))^-1, {i, 1, Length[pmo3]}]}][[1]]
```

Laten we de verhouding voor de eerste 10000 priemgetallen bekijken. Timing[] bepaalt de tijd die *Mathematica* over deze berekening doet:

```
p3p1 = Timing[N[p3[1, 100000]/p1[1, 100000], 20]]

{37.594 Second, 1.4871655814206811459}
```

Mathematica doet er dus ongeveer 37,5 seconden over.
De β-functie is een somrij (net als de ζ-functie), maar niet eentje die *Mathematica* standaard kent. Ik moet hem nog even definiëren:

```
DB[x_, k_] := Sum[(-1)^n/(2n + 1)^(x), {n, 0, k}]

BLIM[n_] :=
  Module[{x = n, expr = 1},
    While[x != -1,
      expr = Sqrt[(1 - 2^(-2^(x + 1)))*
            Zeta[2^(x + 1)]/(DB[2^x, \[Infinity]])^2*expr]; x = x - 1]; expr]
```

Eerste een tabel met de benaderingen voor n = {1, 2, 3, 4, 5}, daarna met het verschil tussen de benaderingen en de boven berekende waarde ervan (met 10000 priemgetallen).

```
benadering = Table[Timing[N[BLIM[i], 20]], {i, 1, 5}]
```

```
{{0.032 Second, 1.4830557664224863250}, {0.046 Second,
    1.4872121328716638225}, {0.094 Second,
    1.4872400244256083148}, {0.125 Second,
    1.4872400265843418256}, {0.188 Second, 1.4872400265843418507}}
```

Zelfs de beste benadering duurt maar 0,2 seconden! Maar hoeveel is de afwijking:

```
Table[benadering[[i]][[2]] - p3p1, {i, 1, 5}]
```

```
{-0.0041098149981948210, 0.0000465514509826766,
0.0000744430049271689, \ 0.0000744451636606797,
0.0000744451636607048}
```

Mijn benadering is zo veel beter dan de brute-force methode van priemgetallen proberen dat hij er met n = 2 al veel dichter bij zit. En dat ongeveer 150 keer zo snel.

2.9 Mogelijke generalisaties

In dit geval werkten we met het Number Theoretic Character χ_4 en de daarbij behorende L-series. Het is mogelijk de oplossing uit te breiden naar andere Number Theoretic Characters. Zo zou voor de functies ψ_5 en ψ_1, waarbij het Number Theoretic Character χ_6 hoort, het volgende gelden:

$$\psi 1[s] = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

voor p alle priemgetallen die modulo 6 gelijk zijn aan 1

$$\psi 5[s] = \prod_q \frac{1}{\left(1 - \frac{1}{q^s}\right)}$$

voor q alle priemgetallen die modulo 6 gelijk zijn aan 5
Het is dus mogelijk om $\psi 5[1]/\psi 1[1]$ als volgt te benaderen:

1. Kies een natuurlijk getal n.
2. Benader $\frac{\psi 5[2^n]}{\psi 1[2^n]}$ door te stellen $\frac{\psi 5[2^n]}{\psi 1[2^n]} = 1$
3. Gebruik de volgende recursieve formule meerdere malen om uiteindelijk op $\frac{\psi 5[1]}{\psi 1[1]}$ uit te komen:

$$\frac{\psi 5\left[2^{n-1}\right]}{\psi 1\left[2^{n-1}\right]}=\sqrt{\frac{\left(1-\frac{1}{2^{2^n}}\right)\left(1-\frac{1}{3^{2^n}}\right)\zeta\left[2^n\right]}{L_6\left[2^{n-1},\chi_6\right]^2}\frac{\psi 5\left[2^n\right]}{\psi 1\left[2^n\right]}}$$

De extra term met de 3 komt door het feit dat het Number Theoretic Number χ_6 nu zowel 2 als 3 niet meeneemt (beiden zijn modulo 6 niet gelijk aan 1 of 5), terwijl χ_4 eerst alleen 2 wegliet.