Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 April 2021**. The solutions of the problems in this issue will appear in the next issue.

Problem A (proposed by Daan van Gent and Hendrik Lenstra)

Let G and A be groups, where G is denoted multiplicatively and where A is abelian and denoted additively. Assume that A is 2-torsion-free, i.e. it contains no element of order 2.

Suppose that $q : G \to A$ is a map satisfying the parallelogram identity: for all $x, y \in G$ we have

$$q(xy) + q(xy^{-1}) = 2q(x) + 2q(y).$$

Prove that for all $x, y \in G$ we have $q(xyx^{-1}y^{-1}) = 0$.

Problem B (folklore)

Prove that every Jordan curve (i.e. every non-self-intersecting continuous loop in the plane) contains four points A, B, C, D such that $ABCD$ forms a rhombus.

Problem C (proposed by Daan van Gent)

A *directed binary graph* is a finite vertex set V together with maps $e_1, e_2 : V \to V$. (The edges are formed by the ordered pairs $(v, e_i(v))$ with $i \in \{1,2\}$.)

For $a,b,c,d \in \mathbb{Z}_{>0}$, an $(a:b)$*-to-*$(c:d)$ *distributive graph* is a directed binary graph G together with distinct vertices $s, t_1, t_2 \in V$ such that G interpreted as a Markov chain has the following properties:

1. For all $v \in V$ the edges $(v, e_1(v))$ have transition probability $\frac{a}{a+b}$ and edges $(v, e_2(v))$ have probability $\frac{b}{a+b}$.
2. It has the initial state s with probability 1.
3. Both t_1 and t_2 connect to themselves, meaning $e_i(t_j) = t_j$ for all $i, j \in \{1,2\}$.
4. It has a unique stationary distribution of t_1 with probability $\frac{c}{c+d}$ and t_2 with probability $\frac{d}{c+d}$.

Show that for all $a,b,c,d \in \mathbb{Z}_{>0}$ there exists an $(a:b)$-to-$(c:d)$ distributive graph.

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