Problemen

| Problem Section

Problem A (folklore)

For a finite sequence $s=(s_1,\ldots,s_n)$ of positive integers, denote by $p(s)$ the number of ways to write s as a sum $s=\sum_{i=1}^{n} a_i e_i+\sum_{j=1}^{n-1} b_j(e_j+e_{j+1})$ with all a_i and b_j non-negative. Here e_i denotes the sequence of which the i-th term is 1 and of which all the other terms are 0.
Show that there exists an integer $B>1$ such that for any product F of (positive) Fibonacci numbers, there exists a finite sequence $s=(s_1,\ldots,s_n)$ with all $s_i\in\{1,2,\ldots,B\}$ such that $p(s)=F$.

Problem B (folklore)

Let ℓ be a prime number. For any group homomorphism $f:A\to B$ between abelian groups and for any integer $n\geq 0$, denote by f_n the induced homomorphism $A/\ell^n A\to B/\ell^n B$. Let $(k_n)_{n=0}^{\infty}$ and $(c_n)_{n=0}^{\infty}$ be sequences of integers.
Show that there exist integers $N,a,b\geq 0$ and a group homomorphism $f:(\mathbb{Z}/\ell^N\mathbb{Z})^a\to(\mathbb{Z}/\ell^N\mathbb{Z})^b$ such that for all $n\geq 0$ we have $\#\ker f_n=\ell^{k_n}$ and $\#\operatorname{coker} f_n=\ell^{c_n}$ if and only if $k_0=c_0=0$ and the sequences $(k_{n+1}-k_n)_{n=0}^{\infty}$ and $(c_{n+1}-c_n)_{n=0}^{\infty}$ are non-negative, non-increasing, eventually zero, and there is a constant C such that for all n such that $k_{n+1}-k_n$ and $c_{n+1}-c_n$ are not both zero, their difference is C.
(Recall that the *cokernel* $\operatorname{coker} f$ of a group homomorphism $f:A\to B$ between abelian groups is the quotient of B by the image of f.)

Problem C (folklore)

Let R be the polynomial ring over $\mathbb{Z}$ with variables x_i, y_i, z_i for all $i\in\mathbb{Z}$. Let S be the polynomial ring over $\mathbb{Z}$ with variables t_i for all $i\in\mathbb{Z}$. Let $\tau:R\to R$ be the isomorphism of rings given by $x_i\mapsto x_{i+1}$, $y_i\mapsto y_{i+1}$ and $z_i\mapsto z_{i+1}$.
Consider the morphism $f:R\to S$ of rings given by $x_i\mapsto t_{i-1}t_it_{i+1}$, $y_i\mapsto t_i^3$ and $z_i\mapsto t_i^2$. Does there exist a finite number of elements $r_1,\ldots,r_n\in R$ such that the kernel I of f is generated as an ideal in R by $\{\tau^i r_j: i\in\mathbb{Z},\ j=1,\ldots,n\}$?

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